UNIVERSIDAD AUTÓNOMA DE SAN LUIS POTOSÍ

FACULTAD DE CIENCIAS

SIMULTANEOUS FIRST-PRICE AUCTIONS FOR COMPLEMENTARY GOODS AND SOME APPLICATIONS.

TESIS

PARA OBTENER EL GRADO DE:

DOCTORA EN CIENCIAS APLICADAS

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SAN LUIS POTOSÍ, S.L.P. ABRIL 2023

Declaración de autoría y originalidad de la tesis.

Yo, Karla Flores Zarur, estudiante del Posgrado en Ciencias Aplicadas de la Facultad de Ciencias de la Universidad Autónoma de San Luis Potosí, como autora de la tesis "Simultaneous Auctions for Complementary Goods and Some Applications", declaro que la tesis es una obra original, inédita, auténtica, personal, que se han citado las fuentes correspondientes y que en su ejecución se respetaron las disposiciones legales vigentes que protegen los derechos de autor y de propiedad intelectual e industrial. Las ideas, doctrinas, resultados y conclusiones a los que he llegado son de mi absoluta responsabilidad.

Resumen

En esta tesis mostramos diferentes Equilibrios Bayesianos Simétricos para un contexto de m subastas simultáneas cerradas del primer precio y n postores por bienes complementarios. El Capítulo 3 consideramos que las valoraciones individuales de los m diferentes bienes son de conocimiento común e idénticas entre los individuos y, de llegarse a obtener el conjunto completo de bienes por un mismo comprador, un valor privado e independientes es obtenido por éste individuo. En el Capítulo 4, analizamos un problema de m subastas simultaneas cerradas del primer precio por bienes complementarios idénticos. Adicionalmente ofrecemos un análisis del rendimiento esperado del vendedor para cada caso. Finalmente, en el Capítulo 5 desarrollamos dos aplicaciones de juegos Bayesianos aplicado al contexto de Cadenas de Suministro.

Abstract

This thesis shows different Symmetrical Bayesian Nash Equilibrium in a context of m simultaneous first-price sealed-bid auctions and n bidders for complementary goods. In Chapter 3, we consider that the individual valuations of the m different goods are common knowledge and identical among bidders and if the whole set of goods is gained by the same buyer, a private independent extra profit is obtained by this buyer. In Chapter 4, we develop a problem where we have m simultaneous first-price sealed-bid auction, for identical complementary items. On addition, we provide an analysis of the seller expected revenue for each case. Finally, in Chapter 5 we develop two applications of Bayesian games in supply chains.

Este trabajo está dedicado
a la persona que inspira,
motiva y llena mi vida de alegria:

Georgette.

Agradecimientos

Primero que nada quiero agradecer profundamente a Dios por haberme permitido llegar hasta aquí y por siempre haber llenado mi camino de bendiciones.

Después, quiero agradecer a las personas a quienes debo mi existencia: mis padres. Mamá, gracias por siempre estar dispuesta a apoyar en todo. Tu me diste la enseñanza de la generosidad y la perseverancia. Papá, gracias por siempre haber creído en mí. Tu me enseñaste el camino de la rectitud y la disciplina. Gracias a ambos por todo.

Así mismo, quiero expresar mi gratitud al Dr. William José Olvera López quien para mi fue mas que un asesor de tesis, un mentor. Sus aportes en mi vida no solo fueron académicos si no también de mucho crecimiento personal. Simplemente, gracias profesor.

De igual manera, deseo expresar mi agradecimiento a mis revisores, no solo por sus muy valiosas aportaciones en el presente trabajo si no también por haber dado el seguimiento adecuado a mi trabajo desde un inicio.

En tercer lugar, quiero expresar mi gratitud al Consejo Nacional de Ciencia y Tecnología (**CONACyT**) por haber financiado mis estudios doctorales durante cuatro años.

También quisiera extender mi agradecimiento a mi amiga Claudia Motilla quien siempre nos brindo su sincera amistad tanto a mi hija como a mí durante este largo proceso donde los retos no suelen ser únicamente de indole académica sino también en gran medida de indole personal. Gracias por todo tu apoyo.

Quiero agradecer de manera muy especial a la familia Flores Medina quienes siempre me han brindado un cariño muy sincero, apoyo incondicional y atenciones verdaderamente valiosas especialmente con mi hija. Gracias por tanto.

Finalmente, quiero agradecer a mis amigos y compañeros del instituto quienes me han visto desarrollarme en este ámbito y han hecho de este un viaje muy agradable: Iván Téllez, Joss Sánchez, Julio Macías, Rafa Alcaraz, Gallo Ugalde, Felipe García, Carlos Reyes, Andrea España, Andrea Alva y Luz Santos. Gracias por todo el apoyo y la amistad brindada.

Contents

Chapter 1

Introduction

People constantly interact in societies, economies, politics, and almost every situation involves more than one person. The importance of understanding how people behave and how they should be advised in strategic situations lets us anticipate the possible ways in which a situation can result. **Game Theory** offers a full spectrum of tools to analyze almost all kinds of interactive situations, cooperative and non-cooperative, where a person's behavior affects all other people involved and the final outcome depends on the joint actions.

The seminal work which starts with this theoretical mathematical analysis in 1944 is John Von Neumann and Oskar Morgenstern, "Game Theory and Economic Behavior". Later, in 1953, John Nash proposed a more general way to model a game considering a different strategy space and proposed the most important solution for games, known as the *Nash Equilibrium.* Decades from these works, Game Theory has very important implications on design, analysis, and execution of all kinds of mechanisms oriented to many different proposals like pricing, public politics applications, cost and gain determination, bargaining, voting, fair division, matching, and so many others.

Particularly, situations where people compete for some objective, as in a contest for a job or buying a unique piece of art, are contexts that concern *non-cooperative* Game Theory. In this Thesis, we will focus on these interactions where we try to understand how agents, called *players*, behave, and how they should be advised in strategic situations, called *games.* We do this by providing a formal framework and a unified language that makes it possible to describe and analyze specific contexts and discuss ideas about the agents' behaviour.

In non-cooperative Game Theory, we can find different ways to construct models with strategic settings depending on the availability of the information for the players. The latter refers to the knowledge that every agent has concerning with the actions that they and others can take, the previous actions that others or themselves took in the past, probabilistic situations, and in general, every information piece that affects the performance of the game. Contexts, where agents do not have all the information relevant to the decision problem are usually called *incomplete information games*. An example of these kinds of situations is where players do not have relevant information concerning other players' preferences about a disputed asset like in the case of the auctions which is the main subject of this thesis jointly with the study of some supply chain applications in similar incomplete information contexts.

Bayesian Games are the proper tool for analyzing any incomplete information game. These games were proposed first in 1967 by John Harsanyi, a Hungarian-American economist that received the Nobel Memorial Prize in Economic Sciences in 1994. The study of auctions is focused on the strategic behavior of bidders and sellers, that is, how to bid according to the information that bidders have and how much income to expect for the seller's case according to the implemented mechanism. The analysis of bidder's strategic behaviour, and seller's revenue in auctions, dates from 1961 and is attributed to William Vickrey for his work named "Counterspeculation, auctions, and competitive sealed tenders", although other authors studied auctions before from different perspectives. Also, studying the allocation mechanism, that is to say, the rules under which an object is assigned to an agent, is an important approach in auction analysis. There are so-many different mechanism to allocate an item through an auction, but in this Thesis, we will focus our attention on the well-known *first-price sealed-bid* mechanisms due to the theoretical importance and the simplicity in which this mechanism is applied in a real context.

As we introduced before, auctions as Bayesian games consider that bidders do not have accurate information about how the auctioned objects are valued by their opponents. Classically, this lack of information is modeled by independent and identically distributed random variables associated with the individual valuation of the auctioned object. This assumption represents a kind of symmetry regarding the inaccurate information among bidders and the fact that each bidder's valuation cannot be influenced by how the object is valuated by other agents. This kind of model has been successfully studied by Vickery (1962) as well as Milgrom and Weber (1982) and more recently by Young, H. Peyton, and Shmuel Zamir (2015); all of these studies consider symmetry in bidding strategies which depend on the value of the item. Lorentziadis, P. L. (2016) includes asymmetry in strategic bidding analysis and includes other approaches like the mechanism design.

As we can see, auctions are a very well-studied subject when the analysis focuses on one single auction for one single object. When we reformulate the situation as a problem of simultaneous auctions for complementary goods, where the first-price sealed-bid mechanism is preserved, results in the literature regarding strategic behavior become scarce, notwithstanding the multiple applications within electronic commerce and other areas. Krishna and Rosenthal

(1996) reach some results regarding to the strategic behavior in a framework of second-price simultaneous auctions with synergy. Besides, Rosenthal and Wang (1996) show necessary and sufficient conditions for a strategy to be a symmetric equilibrium in a particular case of first-price simultaneous auctions. Later, Szentes and Rosenthal (2003) show an equilibrium with interesting geometrical interpretation for a case of three identical items, two bidders, and complete information for a case of complementary goods. Subsequently, Szentes (2007) states a model of two simultaneous auctions, two objects, and two bidders with complete information for the case of complements and substitute goods.

We will board two cases of first-price sealed-bid simultaneous auctions for complementary goods. The theoretical complexity focuses on the fact that the general probabilistic expressions do not allow us to offer standard solutions because there is no direct way to obtain useful expressions for mathematical manipulation. This problem arises because of the complementary nature of the goods, which makes us consider that the bidding functions depend on two variables: the individual valuation of the item and the extra profit one can obtain when having the whole set of items. Thus, comparing bidding functions of two or more variables is almost impossible without making assumptions. For example, consider a situation where two bidders participate in two simultaneous auctions. In the former, a seller is auctioning a very rare sticker album, and in the latter a package of stickers for that album. Each bidder has a personal valuation of the sticker album, the package of stickers, and the whole set of items. It is clear that, in this case, the valuation of the set of items exceeds the sum of the valuations for the individual auctioned items. Thus, with this example, we can conceive that bidders may be tempted to bid above their separate stand-alone valuations of the individual items hoping to win the set as a reasonable way to act, although they would taking a risk of overbidding on items in case of failing to win the whole set. This Thesis proposes solutions for two particular cases of simultaneous auctions for complementary goods and also includes two applications of Bayesian games in a supply chain context.

Also, we analyze the relationships and different ways of interacting and coordinating between the agents involved in any of the phases of production or supply, which is known as the supply chain. A supplier-retailer supply chain, which is a two-link, single-channel supply chain, represents the relation between a manufacturer who produces some kind of merchandise to sell to a retailer who purchases it for the purpose of re-selling to a final consumer. In general, the links represent the different phases in the chain and the channels represent the different options to carry out the same phase of the process. For example, the manufacturer-retailer relationship where there is a single supplier and two retailers would be described as a single-link, two-channel supply chain. This type of model has been successfully studied by Yang and Zhou (2006) considering a context where retailers form a duopoly. Likewise, F. Chen (2003), Chiang et al (2005), among others, have provided interesting studies regarding the analysis of different supply chain models.

Our interest in this subject lies in the importance of the analysis of supply chains to improve the operation of commercial activity and in the increasing interest for corporations to invite people and companies to join as business partners. In the first application, we board the importance of the information exchange in a bargaining problem between a supplier and a retailer using signaling games. In the second application, we analyze a bargaining situation between a supplier and two retailers in a context where it is not possible for the supplier to satisfy his commercial partners' demand.

Thus, this Thesis is divided into four chapters of which a summary is presented below:

- First, we will start with a Preliminaries chapter where we present some classical notation for defining a game, strategies, the solution concept of Nash equilibrium, a Bayesian game, and the Bayesian Nash equilibrium accompanied by some examples for clarifying the definitions.
- In Chapter 3 we develop our first simultaneous auction for complementary goods model. We propose a model where bidders rationalize their bids according to two criteria: their own valuation of each item and an extra profit. We assume that a bid is given by a bidding function that depends on those criteria, say $b(x, y)$; the problem arises when we try to follow the classical methodology and compare bids for calculating the corresponding probabilities. For instance, if we consider the previous sticker album example, and we have that a bidder values the sticker album in zero but the whole set of items in 100, and another bidder values the sticker album in 10 and the whole set of items in 90, how do we should compare the bids given by $b(0, 100)$ with $b(10, 90)$? Even assuming that bidders rationalize the bids in a symmetrical way, according to their own subjective valuations, it is difficult to offer a general way to compare the bids.

  In this chapter, we develop a case where the individual valuations of m different objects are identical among the bidders and this information is common knowledge. Assuming complementarity among the goods, there is a chance of earning a private independent extra profit only in case when the whole set of goods is gained. A particular example is when someone who wants a buildable collectible toy participates in several simultaneous auctions to get each piece but with the particularity that the individual pieces are valued at a standard market price, and the complete set valuation is a private and independent value that exceeds the sum of the valuation of the individual pieces. Nowadays, the above situation often occurs due to the fact that users of electronic commerce platforms acting as sellers face a decision problem about how to choose the selling mechanism of their merchandise. They can choose to auction all the items, try to sell them at a posted price, or put a subset of them up for sale and auction the rest, etc. Hilda Etzion et al (2006) have a result about the optimal choice for the seller when a similar situation occurs.

For solving the problem related before, in this paper we propose a bidding function $b(x,y) = g(x) + h(y)$ as the sum of two increasing and non-negative functions such that $g(0) = h(0) = 0$, where g represents the part of the total bid concerning only to the individual valuation of an item and h represents how much more an agent is willing to bid based on the extra profit. This particular bidding form implies that the change on b, when a criterion changes, only depends on that criterion. The previous assumption allow us to offer a Symmetric Bayesian Nash Equilibrium (SBNE), given by:

$$b_j^i = b(a_j, c_i) = g^*(a_j) + h^*(c_i)$$

where:

$$g^*(a_j) = a_j;$$

$$h^*(c_i) = \frac{(n-1)\int_0^{c_i} xF(x)^{n-2}f(x)dx}{mF(c_i)^{n-1}}.$$

The way in which we provide the proof is constructive, and it ensures that our candidate is, in fact, an SBNE. Additionally, we have that under our equilibrium, a bidder with the highest valuation over the whole set of items will gain the complete set, ensuring that no bidder has a negative profit even if they bid above his own individual valuation. Also, Section 3.2 shows that under this equilibrium, the seller's expected profit when he performs simultaneous auctions and a single first-price sealed-bid auction for the whole set of items is the same. Finally, Section 3.3 shows some remaining open questions and the main challenges on this topic.

Previous results have been summited and subsequently accepted for their publication in the *Journal of Dynamics and Games*.

- In Chapter 4 we develop a problem where we have m simultaneous first-price sealed-bid auction, for identical complementary items. An example of this problem can be observed in the electronics industry when some companies try to carry out products under special conditions that can be sold as brand-new products. These special products have been passed for different processes like repair or reassembly, or they were simply removed from de original packaging and returned. Subsequently, companies usually allocate such products by auctioning lots of identical goods through special e-commerce pages where retailers look for good-price merchandise. We see that in this case, potential buyers are interested in a complete lot because they are not consumers and they want the items for re-selling. We assume that each potential buyer has an individual and independent valuation for an item and, for his part, the seller has a lot of items that have to decide how to sell. Thus, we study two different ways to sell a lot of identical items: first, through m simultaneous first-price sealed-bid auctions where if one bidder wins the whole lot the seller gives him a bonus. Second, selling a whole lot of items (bonus included) by one first-price sealed-bid auction.

From the simultaneous auction case, we study if higher income for sellers can be obtained through the different ways to offer a bonus. We propose that the bonus is given as a function of the individual valuation of the object according to a function $H(\cdot)$, which could be interpreted as free delivery, extra pieces, an amount of money, etc. Thus, we see that in this case, the valuation of the complete set of items is greater than the sum of the individual valuations of the objects for all the bidders. Under this context, we propose that bidders rationalize their bids according to the valuation of a single object and the extra utility given by the bonus. The previous framework will allow us to take this situation as a Bayesian game considering the individual valuation as a private and independent value and, find the equilibria associated with the strategic behavior of the bidders when they rationalize their bid symmetrically according to a bivariate function $b(a_i, H(a_i))$. For this case, we obtain the equilibrium bidding function that follows:

$$b_i = b(a_i, H(a_i)) = \alpha^*(a_i)$$

where:

$$\alpha^*(a_i) = \frac{(n-1)\int_0^{a_i} xF(x)^{n-2}f(x)dx}{F(a_i)^{n-1}} + \frac{(n-1)\int_0^{a_i} H(x)F(x)^{n-2}f(x)dx}{mF(a_i)^{n-1}}.$$

For more details see Section 4.1. Additionally, in Subsection 4.1.1 we checked if the expected return of the seller is affected by different proposals for the function $H(\cdot)$ and we compare our simultaneous auction case with the situation where the seller decides to auction a complete lot of objects together with the bonus through a single closed auction of the first price. Surprisingly we conclude that the seller's expected return in both cases is the same.

The results of this chapter have been summited and subsequently accepted for their publication by the *EconoQuantum* journal.

- Finally, in Chapter 5 we develop two applications of Bayesian games in supply chains.

  First, in Section 5.1 we model a supply chain that considers only one supplier and one retailer that interact and interchange market demand information. In this proposal, we are focused on information exchange considering that the agents' utility functions take into account variables like production quantity, ordering policy, purchase quantity commitment, the cost of item breakage, and stock-out costs. The interaction between the supplier and the retailer can be broadly classified into three groups based on the following assumptions with respect to the market demand: the interaction is governed by constant demand, the seller and the buyer independently study how demand varies, and the interaction between the seller and the buyer favors the quality of the information regarding the demand. In this application, we will study the latter case, specifically we will analyze the case that assumes that the retailer has better market information since he is closer to the market.

  Thus, we analyze this situation from the **Signaling Game** perspective, proposing that the agents' utility functions are similar to that proposed in I. Slimani and S. Achchab's (2014) supply chain model because they consider the variables that we mentioned before in a simple and general way. For simplicity, we use the basic structure of a signaling game and consider that market demand is a discrete random variable with two possible realizations: *low* and *high* demand. We develop two models under different assumptions regarding the bargaining policies between agents and find out that rational behaviour does not necessarily implies cooperation between agents unless we consider policies that promote this cooperation, this means that an equilibrium behaviour can imply that agents lie or do not follow a signal. For more details of a model that implies non-cooperative actions as part of a rational behaviour refer to Section 5.1, and for more details about a model where rational behaviour is always side by side with cooperation, see Section 5.1.1.

  This results have been summited and subsequently accepted for their publication in the *Cuadernos de Economía* journal.

  Secondly, in Section 5.2 we develop a model that involves one supplier and two retailers competing for the seller's production lot. This is a situation where is known that the complete supplier's production lot is not enough to satisfy individual demands from the two commercial partners. We focus on proposing auctions as an allocation mechanism that ensures for a seller to satisfy completely one agent's requirement, partially satisfy the other agent's requirement, and guarantees a better-expected outcome compared with selling the whole lot to a given price, say $\hat{p}$, making an arbitrary allocation. We propose that a bid is composed by two information pieces: the *true* requirement and the unitary bid that a bidder is allowed to pay in case his unitary bid where the highest one. Thus, our mechanism only takes into account the unitary bid as a rule to allocate the items according to their magnitude. Additionally, we consider that buyers face a stock-out cost that affects symmetrically both players. This symmetry could be the case of considering similar commercial partners that are constituted in a similar way and face the same consequences when they can not satisfy their own clients' demands. We model the previous situation as a Bayesian game where it is supposed that the other agent's requirement and the valuation for one single item are independent random variables. Additionally, we assume that players bid symmetrically according to an increasing function $b(\cdot)$ that depends on one item individual valuation. Thus, we fix the problem from player one perspective and assume that player 2 bids according to his own item's personal valuation. Under these assumptions, we were able to find for this case an equilibrium bid given by:

$$b^*(v_1) = \left(\frac{q_1+1}{2q_1}\right)\left(v_1 + \alpha - \frac{\int_0^{v_1} F(x)dx}{F(v_1)}\right) + \left(\frac{q_1-1}{2q_1}\right)\hat{p}.$$

For more details of this construction sees Section 5.2. We have to mention that this work has not been sent to a scientific journal yet, so the work could be modified according to the comments of some referees. For a discussion of this case see Subsection 5.2.1.

Chapter 2

Preliminaries

People used to describe a game as a situation where the participants lose or win something according to some rules as sports or leisure games. Nevertheless, this description is limited to analyzing interactive behavior and the consequences for the agents involved. Instead of that notion for describing a game, we will focus on defining a game as a precise and logically consistent structure describing a strategic setting associated with a situation. We will identify a set of players, a complete description of the actions they can do, a description of players' information about other players' moves in every possible part of the game, how the players' actions lead to outcomes, and a description of players' preferences over the possible outcomes. In this chapter, we will provide a formal definition of a game and the way we will use it to analyze decision-making problems.

One possible way to define a game is in its extensive form, which is useful to analyze situations with many stages, as in the game of chess. The extensive form is also useful to model situations with stochastic moves like the roll of dice as well as to model contexts with *imperfect information*. The latter means that some agents have to make a decision without knowing the precise combinations of movements in previous stages.

The concept of a strategy of an agent will be taken as a complete description of what a player does every time he has to make a decision in the game, even if one or more of those moments are exclusive from each other. We will denote the player i's strategy set by S_i and a particular player i's strategy (contingency plan) by s_i. Thus, we define a **strategy profile set**[1] denoted by $S = S_1 \times S_2 \times ... \times S_n$ containing all possible ways to perform a game.

[1]The set can be finite or infinite depending on the model. The generalization from the finite strategy profile set definition to the infinite profile set, is direct, although some classical results could depend on this feature.

There is an alternative way to define a game, its normal form. The normal form of a game consists on modeling how a decision-maker chooses a plan of action given that all others choose simultaneously and independently. Under this interpretation, each player is unaware of other players' choices when he chooses an action. The model consists of a finite set of players N, a set of actions A_i for each player $i \in N$, and a utility function u to model the players' preference relation over the possible outcomes.

Definition 2.1. *A* ***normal form game*** *G consists of*

- *a finite set $N = \{1, 2, ..., n\}$, the set of* ***players*** *or the set of agents,*
- *an* ***action profile set*** *$A = A_1 \times A_2 \times ... \times A_n$, where for each player $i \in N$, A_i is a nonempty set of* ***actions*** *available to player i. We say that $a \in A$ is an* ***action profile****,*
- *a utility function $u = (u_1, ..., u_n)$ where, $\forall i \in N$, $u_i : A \to \mathbb{R}$.*

Thus, we denote a game G by a tuple $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$.

Remark. *To refer only to player i's utility we will write $u_i(\cdot)$.*

Remark. *In a normal form game, a player i's strategy s_i is equivalent to a player i's action a_i.*

Remark 2, foregrounds that a normal form game *actions* abridge a complete contingency plan described in a *strategy*. Thus, an action summarises a total description of how the players execute their actions along several stages. Sometimes we refer to a member of A_i as a **pure strategy**.

Remark. *If the set A_i of actions of every player i is finite then the* ***game is finite****.*

A finite normal form game with two players can be conveniently modeled as follows:

		Player 2	
		L	R
Player 1	T	$(3,3)$	$(4,2)$
	B	$(2,4)$	$(2,2)$

Figure 2.1: Game G: A normal form game.

Figure 2.1 represents a game where

$$A_1 = \{T, B\},$$

$$A_2 = \{L, R\},$$
$$A = \{(T, L), (T, R), (B, L), (B, R)\},$$
$$u((T, L)) = (3, 3),$$
$$u((T, R)) = (4, 2),$$
$$u((B, L)) = (2, 4),$$
$$u((B, R)) = (2, 2).$$

Notice that each action profile $a \in A$ describes a different way to *play* the game and the function u describes the utility for every player. Thus, using the notation of Remark 2 the utility for player 1 when the action profile is (T, L) is given by $u_1((T, L)) = 3$.

A central solution concept in Game Theory is **Nash equilibrium**. This concept proposes a solution that implies agents' rationality. Intuitively, if players are aware of their alternatives, they will choose their actions after some optimization process. In this sense, we say that, as a minimal rationality requirement, players always have to know their best responses for a given other players' action profile.

To define this concept properly, we must first propose a notation that allows us to talk about all of the *other players'* actions given a single player i. So, we say $-i$ to refer to those players. Thus, a_{-i} is an action profile for every player except player i, in general we have that $a_{-i} \equiv (a_1, a_2, ..., a_{i-1}, a_{i+1}, ..., a_n)$. To refer to an action profile a distinguishing between player i's action and the other players' action, we write $a = (a_i, a_{-i})$. Thus, we have the next definitions:

Definition 2.2. *$a_i^* \in A_i$ is a **best response** to a_{-i} if and only if, for all $a_i \in A_i$ we have $u_i((a_i^*, a_{-i})) \geq u_i((a_i, a_{-i}))$.*

Definition 2.3. *An action profile $a \in A$ is a **pure Nash equilibrium** (PNE), if and only if, for all $i \in N$, a_i is the best response to a_{-i}.*

Definition 2.3 expresses that some strategy profiles or action profiles have desirable features that characterize players' rational behaviour. Specifically, this idea focuses on the Definition 2.2 which describes that, for all players, given the other players' actions profile, a rational player would choose the alternative that maximizes his utility. In other words, a PNE is a strategy profile such that given the other players' actions, if any player $i \in N$ changes his action, he does not get a better outcome. We can observe an example from the game G in Figure 2.1 when we focus on the strategy profile (T, L): we can see that if Player 1 changes from action T to action B then he worsens his payment and we have a similar consequence for Player 2 if he changes from L to R. As well, if we observe the other strategies profiles $\{(T, R), (B, L), (B, R)\}$, we have that both player 1 and player 2 can always get a better outcome if they change their actions.

Example 2.1. *Each of the two people chooses either Head or Tail. If the choices differ, Player 1 pays Player 2 a dollar; if they are the same, Player 2 pays Player 1 a dollar. A game that models this situation is shown in the next figure:*

		Player 2	
		$Head$	$Tail$
Player 1	$Head$	$(1,-1)$	$(-1,1)$
	$Tail$	$(-1,1)$	$(1,-1)$

Figure 2.2: Game G': Matching Pennies.

Figure's 2.2 game G' shows a situation where the action profile is

$$A = \{(Head, Head), (Head, Tail), (Tail, Head), (Tail, Tail)\}.$$

We can observe that if we pick a strategy from A, no matter which one has been chosen, there is always a way for players to get a better response, so we do not have a PNE for this game. Until this point, we realize that our solution concept from Definition 2.3 has very serious limitations since such strategy does not always exist.

In order to solve the inconvenience of the previous framework we must provide a more general definition of a strategy profile.

Definition 2.4. *Given a game G, a **mixed strategy** σ_i, is a probability distribution over the strategy profile S_i.*

Notation.

- *A **player i's mixed strategy** will be denoted by σ_i.*
- *The **i's mixed strategies set** will be denoted by Σ_i.*
- *The **mixed strategies profile set** $\Sigma_1 \times \Sigma_2 \times ... \times \Sigma_n$ will be denoted by Σ.*
- *A **mixed strategy profile** will be denoted by $\sigma \in \Sigma$.*
- *For any finite set S_i and $\sigma_i \in \Sigma_i$ we denote by $\sigma_i(s_i)$ the **probability that σ_i assigns to** $s_i \in S_i$.*
- *We call the **support of** σ_i the set of elements $s_i \in S_i$ for which $\sigma_i(s_i) > 0$.*

Remark. *A profile $\{\sigma_i\}_{i\in N}$ of mixed strategies induces a probability distribution over the set S.*

Thus, a mixed strategy proposes that in a game the participants' choices are not deterministic but are regulated by probabilistic rules. This means that the decision of a player can be governed by random behaviour. In simpler words, we have that a player $i \in N$ will make a decision according to a probability distribution over the strategy set Σ_i. Often, those kinds of decision-making situations happen in the real world, for example when someone has to decide the color to dress for dinner, or when some government randomly audits taxpayers. Now, we can generalize our first solution concept as follows:

Definition 2.5. *A* ***mixed strategy equilibrium*** *(MSE), σ^*, is a mixed strategy profile such that $u_i(\sigma^*) \geq u_i(\sigma^*_{-i}, \sigma_i)$ for all $\sigma_i \in \Sigma_i$ and for all $i \in N$.*

It would be useful to sort out under which conditions a game has or does not have a Nash equilibrium. We have the next equilibrium existence theorem:

Theorem 1. ***(Nash, 1950)****: Every n-personal finite game has at least one MSE.*

Consider our *Matching Pennies* example from Figure 2.2. We have that for all $i \in \{1, 2\}$ the next strategy:

$$\sigma_i = \left(\frac{1}{2}, \frac{1}{2}\right),$$

is a MSE.

The purpose of this thesis requires us to introduce the concept of a **Bayesian Game**. This concept will allow us to analyze situations where agents have incomplete information. These situations refer to an environment where players do not have all the information relevant to the problem, as it can be when players are not certain about other players' characteristics. For example, when someone is encouraged to get involved in a street fight, this person does not know for sure if his opponent has better or worse abilities than him to fight. Another example can be a soccer coach choosing a strategy for a match without knowing if the opponent will be passive or aggressive on the soccer field.

There are different ways to define a Bayesian Game, depending on the purpose. Harsanyi (1967) defined it as a tool to model situations where players' notions of beliefs and knowledge appear as compensation for the lack of information. In this model players' uncertainty about each other is represented by a set Ω of possible **states of nature**. Each state is a description of all players' relevant characteristics like strategies profiles and the outcome set. Besides the states of nature, we use a set T_i of possible *types* to give a more detailed description of the information that player' have. The next definition will be useful to model the different scenarios and an accurate description of the information that a player could have:

Definition 2.6. *A player i's* ***type*** *$t_i \in T_i$, is a player i's belief profile with respect to Ω and the other players' beliefs profiles that characterize it.*

The types enclose very descriptive information for some given belief. For example, a player i's type t_i describes in a broad sense not only the private information about himself but also his own belief about others' private information. Furthermore, player i's type t_i also encloses a belief about the way that others think about player i private information. Thus, we can think about $T = T_1 \times T_2 \times ... \times T_n$ as the **types profile set**. Establishing differences between elements in T and elements in Ω is important. First, a type profile encodes a bigger amount of information than a state of nature profile, which could be the case that more than one type corresponds to the same state of nature, but this distinction is not always relevant. For example, the classical approach of auctions analysis only considers that the relevant information to encode in a type is the bidders' private subjective valuation of objects and the belief about others' private valuation. So, from this point, we will set $\Omega = T$.

With the purpose of being able to describe a contingency plan for each possible player's type, the next definition will allow us to get a more robust players' strategies description than those we have proposed so far.

Definition 2.7. *A **decision function** $s_i(t_i)$, with $s_i : T_i \to S_i$, indicates player i's strategy $s_i \in S_i$ when his type is $t_i \in T_i$.*

Thus, we can state the next formal definition of a Harsanyi Bayesian Game:

Definition 2.8. *A **Harsanyi Bayesian Game** G^B consist of*

- *a finite set $N = \{1, 2, ..., n\}$, the set of **players** or the set of **agents**,*
- *a **strategy profile** set $S = S_1 \times S_2 \times ... \times S_n$,*
- *a **types profile set** $T = T_1 \times T_2 \times ... \times T_n$,*
- *for each player $i \in N$ a **profit function** $\pi(s_i, s_{-i}; t_i, t_{-i})$ that denotes i's utility given that his type is t_i, that he chooses $s_i \in S_i$ and that the other players follow strategies $s_{-i}(t_{-i}) = \{s_j(t_j)\}_{j \neq i}$,*
- *and for each vector $(t_1, t_2, ..., t_n) \in T$, a **distribution function** $F(\cdot)$ over the types profile.*

Thus, we denote a game G^B by a tuple $\langle N, \{S_i\}_{i \in N}, T, \{\pi(\cdot)\}_{i \in N}, F(\cdot)\rangle$.

The probability distribution $F(\cdot)$ indicates the probabilities attached to each combination of types occurring. Thus, we can transform a game of incomplete information into one of imperfect information. Further, for all $i \in N$ we denote by $F_i(t_{-i}|t_i)$ the probability distribution of types t_{-i} of the players $j \neq i$ given that i knows his type is t_i. That is, player i updates his prior information about the distribution of the other types using Bayes rule upon learning that his type is t_i.

Thus, we have the next definition:

Definition 2.9. *A **Bayesian Nash equilibrium** is a list of decision functions* $(s_1^*(\cdot), ..., s_n^*(\cdot))$ *such that* $\forall i \in N$, $\forall t_i \in T_i$ *and* $\forall s_i \in S_i$*:*

$$\int\limits_{t_{-i}\in T_{-i}} \pi_i(s_i^*, s_{-i}^*; t_i, t_{-i})dF_i(t_{-i}|t_i) \geq \int\limits_{t_{-i}\in T_{-i}} \pi_i(s_i, s_{-i}^*; t_i, t_{-i})dF_i(t_{-i}|t_i).$$

Definition 2.9 applies the Nash equilibrium notion to a situation where players consider a bayesian utility function instead of a utility function. Consider the next example taken from Watson, J. (2002):

Example 2.2. *Consider a simple Cournot duopoly game with incomplete information. Suppose that the inverse market demand is given by* $p = 10 - Q$*, where* Q *is the total quantity produced in the industry. Firm 1 selects a quantity* q_1*, which it produces at zero cost. Firm 2's cost of production is private information (selected by nature). With probability* $\frac{1}{2}$*, firm 2 produces at zero cost. With probability* $\frac{1}{2}$*, firm 2 produces at a marginal cost of 4. Call the former type of firm 2 "L" and the latter type "H" (for low and high cost, respectively). Firm 2 knows its type, whereas firm 1 knows only the probability that L and H occur. So we have that* $F_1(L|L) = \frac{1}{2}$ *and* $F_1(H|L) = \frac{1}{2}$ *considering only one type for firm 1 called "L". Let* q_2^H *and* q_2^L *denote the quantities selected by the two types of firm 2. Then when firm 2's type is L, its payoff is given by*

$$\pi_2^L(q_2^L, q_1, L, L) = (10 - q_1 - q_2^L)q_2^L.$$

When firm 2's type is H, its payoff is

$$\pi_2^H(q_2^H, q_1; H, L) = (10 - q_1 - q_2^H)q_2^H - 4q_2^H.$$

As a function of the strategy profile $(q_1; q_2^L, q_2^H)$*, firm 1's payoff is*

$$\pi_1(q_1, q_2^L; L, L)F_1(L|L) + \pi_1(q_1, q_2^H; L, H)F_1(H|L)$$

$$= \frac{1}{2}(10 - q_1 - q_2^L)q_1 + \frac{1}{2}(10 - q_1 - q_2^H)q_1$$

$$= \left(10 - q_1 - \frac{q_2^L}{2} - \frac{q_2^H}{2}\right)$$

Note that firm 1's payoff is an expected payoff obtained by averaging the payoffs of facing the low and high types of firm 1, according to the probability of these types.

To find the Bayesian Nash equilibrium of this market game, consider the types of player 2 as separate players. Then find the best response functions for the three player types and determine the strategy profile that solves them *simultaneously. The best-response functions are calculated by evaluating the following derivative conditions:*

$$\frac{\partial \pi_1}{\partial q_1} = 0,$$

$$\frac{\partial \pi_2^L}{\partial q_2^L} = 0$$

and,

$$\frac{\partial \pi_2^H}{\partial q_2^H} = 0.$$

This yields:

$$q_1^* = 5 - \tfrac{q_2^L}{4} - \tfrac{q_2^H}{4} \text{ for player 1,}$$
$${q_2^L}^* = 5 - \tfrac{q_1}{2} \text{ for player 2 type } L, \text{ and}$$
$${q_2^H}^* = 3 - \tfrac{q_1}{2} \text{ for player 2 type } H.$$

Solving the associated system of equations, the Bayesian Nash equilibrium is found to be the profile $q_1^* = 4, {q_2^L}^* = 3, {q_2^H}^* = 1$. *In words, firm* 1 *produces* 4, *whereas firm* 2 *produces* 3 *if their cost is low and high respectively.*

As we have seen we can use Bayesian games to model many different kinds of incomplete information situations. In this thesis, we are particularly interested in modeling Auctions as Bayesian Games for exploring rational behavior from bidders.

In the following chapters, we will keep the notation defined above to define an auction as a Bayesian game. For example, for defining a first-price auction for one object we have a set of bidders $N = \{1, 2, ..., n\}$, $T_i = [0, \overline{v}]$ the set of possible types of player $i \in N$, and v_i the type received by player i that represent his or her private valuation of the object. We will denote $F(\cdot) : [0, \overline{v}]^n \rightarrow [0, 1]$ as the joint distribution of types and the associated density is denoted by $f(\cdot) : [0, \overline{v}]^n \rightarrow \mathbb{R}_+$. The set of possible bids or strategies for player $i \in N$, is $S_i = \mathbb{R}_+$. As we have seen, under this modeling, any bidder who has incomplete information about other buyers' values is treated as if he is uncertain about their types.

Thus, in the next chapter, we will present a problem of simultaneous auctions for complementary different goods. In chapter four, we will work on a similar problem but assuming identical goods. In chapter five we offer some applications in the field of supply chains using signal games and auctions.

Chapter 3

Simultaneous Auctions For Complementary Goods and Quasi-linear Bids

This chapter shows a Symmetrical Bayesian Nash Equilibrium in a context of m simultaneous first-price sealed-bid auctions and n bidders for complementary goods. We consider that the individual valuations of the m goods are common knowledge and identical among bidders and if the whole set of goods is gained, a private independent extra profit is obtained by the winner. For relaxing and solving the so-many mathematical complications involved in the general case we followed a classical methodology and proposed a particular bidding function that implies linear separability. Under these assumptions, we obtain a Symmetric Bayesian Nash Equilibrium whose functional form implies the classic quasi-linear property for bivariate functions. In addition, we compare the seller expected revenue between auctioning the complete set in one single first-price sealed-bid auction versus auctioning each item in m simultaneous first-price sealed-bid auctions.

3.1 The model.

Let $N = \{1, 2, ..., n\}$ be a set of bidders and $M = \{1, 2, ..., m\}$ a set of items. Each bidder, $i \in N$, has an individual valuation for each item $j \in M$, and it will be denoted by a_j^i. In this section, we study the situation where bidders know their individual values for each item and that opponents have the same valuations as well. This means that for every $k, l \in N$ and for all $j \in M$ we have $a_j^k = a_j^l = a_j$. On the other hand, each player $i \in N$ has an individual independent value denoted by c_i that represents the extra profit that he gains by obtaining all goods. Thus, bidder i's valuation for the complete set of items is $v_i \equiv a_1 + a_2 + ... + a_m + c_i$.

Under this framework, we model a situation where bidders face m simultaneous first price sealded-bid auctions as a Bayesian Game where, for some $\bar{c} \in \mathbb{R}$, the extra profit is chosen independently by cumulative distribution function $F(\cdot)$ with density $f(\cdot) > 0$ in an interval $[0, \bar{c}]$ and it is a private value for each bidder.

We assume that all players follow a bidding strategy $b(\cdot)$. Thus, the game will be analyzed from the point of view of one of the bidders, says bidder 1. Knowing the individual valuations, his extra profit value, and the distribution of the extra profit valuations of bidders $2, ..., n$, bidder 1 has to figure out what should be his best response. The bid of player $i \in N$ for item $j \in M$ will be denoted by b_i^j. Thus, $\{b_1^j\}_{j \in M}$ represents the set of bids from the bidder 1.

Notice that each bidder can obtain 2^m possible subsets of items as the result of the auctions, and therefore, excluding the case where all auctions are lost, there are $2^m - 1$ terms weighted by their probabilities, relevant to the expected utility of bidder 1. Thus, the general expression for bidder 1's expected utility is given by:

$$\begin{aligned}
&\pi(\{b_1^j\}_{j \in M}) = \\
&\sum_{J \subset M} \left[\sum_{j \in J} (a_j - b_1^j) P \left(\left\{ \max_{i \in N \setminus \{1\}} \{b_i^j\} < b_1^j \right\}_{j \in J}, \left\{ \max_{i \in N \setminus \{1\}} \{b_i^j\} > b_1^j \right\}_{j \in M \setminus J} \right) \right] \\
&\qquad + \left(\sum_{j \in M} (a_j - b_1^j) + c_1 \right) P \left(\left\{ \max_{i \in N \setminus \{1\}} \{b_i^j\} < b_1^j \right\}_{j \in M} \right). \qquad (3.1)
\end{aligned}$$

If the term $(a_j - b_1^j)$ is factorized from Equation 3.1, then we can note that the probabilistic term associated to this factor is given by the marginal probability of the random variable $\max_{i \in N \setminus \{1\}} \{b_i^j\}$. For example, if we consider a situation where $m = 3$, and we factorize term $(a_1 - b_1^1)$ from the expected utility of bidder 1 we have simplified expression as follows:

$$
\begin{aligned}
&(a_1 - b_1^1)P\left(\max_{i\in N\setminus\{1\}}\{b_i^1\} < b_1^1\right) = \\
&\quad (a_1 - b_1^1)[P\left(\max_{i\in N\setminus\{1\}}\{b_i^1\} < b_1^1, \max_{i\in N\setminus\{1\}}\{b_i^2\} < b_1^2, \max_{i\in N\setminus\{1\}}\{b_i^3\} < b_1^3\right) \\
&\qquad + P\left(\max_{i\in N\setminus\{1\}}\{b_i^1\} < b_1^1, \max_{i\in N\setminus\{1\}}\{b_i^2\} > b_1^2, \max_{i\in N\setminus\{1\}}\{b_i^3\} < b_1^3\right) \\
&\qquad + P\left(\max_{i\in N\setminus\{1\}}\{b_i^1\} < b_1^1, \max_{i\in N\setminus\{1\}}\{b_i^2\} < b_1^2, \max_{i\in N\setminus\{1\}}\{b_i^3\} > b_1^3\right) \\
&\qquad + P\left(\max_{i\in N\setminus\{1\}}\{b_i^1\} < b_1^1, \max_{i\in N\setminus\{1\}}\{b_i^2\} > b_1^2, \max_{i\in N\setminus\{1\}}\{b_i^3\} > b_1^3\right)]
\end{aligned} \tag{3.2}
$$

Thus, rewriting the general expression given by 3.1 we have:

$$
\begin{aligned}
\pi(\{b_1^j\}_{j\in M}) = \sum_{j\in M}\left[(a_j - b_1^j)P\left(\max_{i\in N\setminus\{1\}}\{b_i^j\} < b_1^j\right)\right] \\
+ c_1 P\left(\left\{\max_{i\in N\setminus\{1\}}\{b_i^j\} < b_1^j\right\}_{j\in M}\right).
\end{aligned} \tag{3.3}
$$

Now, we propose that every bidder rationalizes his bid for each item according to a bivariate function that considers only two criteria. Additionally, we assume that the other agents, aside from player 1, bid according to the individual value of the good and the extra profit value. This means that bidders $i \in N \setminus \{1\}$ bid $b_i^j = b(a_j, c_i)$ with $b(\cdot)$ an increasing function in both variables. Given those conditions, bidder 1 has to find his best response for this situation, which is equivalent to choosing $\{x_j\}_{j\in M}$ and y such that maximizes his expected utility:

$$
\begin{aligned}
\pi(\{x_j\}_{j\in M}, y) = \sum_{j\in M}\left[(a_j - b(x_j, y))P\left(\max_{i\in N\setminus\{1\}}\{b(a_j, c_i)\} < b(x_j, y)\right)\right] \\
+ c_1 P\left(\left\{\max_{i\in N\setminus\{1\}}\{b(a_j, c_i)\} < b(x_j, y)\right\}_{j\in M}\right).
\end{aligned} \tag{3.4}
$$

For this particular case of auctions, it is a desirable property for a bidding function that the change to it, when a criterion changes, only depends on that criterion. A general way to stage that property is by proposing a function $b(\cdot)$ as a decomposition in the sum of two increasing and non-negative functions denoted by g and h such that $g(0) = h(0) = 0$, where g represents the part of the total bid concerning only to the individual valuation and h represents how much more is willing to bid based on the extra profit c_i. Thus, assuming that $b(x, y) = g(x) + h(y)$ we have:

$$\begin{aligned}
&\pi(\{x_j\}_{j\in M}, y) = \\
&\sum_{j\in M}\left[(a_j - g(x_j) - h(y))P\left(\max_{i\in N\setminus\{1\}}\{g(a_j) + h(c_i)\} < g(x_j) + h(y)\right)\right] \\
&+ c_1 P\left(\left\{\max_{i\in N\setminus\{1\}}\{g(a_j) + h(c_i)\} < g(x_j) + h(y)\right\}_{j\in M}\right).
\end{aligned} \tag{3.5}$$

With the previous construction, we can state our first result.

Theorem 2. *For m simultaneous first-price sealed-bid auctions and n bidders with common knowledge on individual valuations, private independent extra profit and bidding strategy given by $b(x, y) = g(x) + h(y)$,*

$$b_j^i = b(a_j, c_i) = g^*(a_j) + h^*(c_i)$$

where:

$$g^*(a_j) = a_j;$$

$$h^*(c_i) = \frac{(n-1)\int_0^{c_i} xF(x)^{n-2}f(x)dx}{mF(c_i)^{n-1}},$$

is a Symmetric Bayesian Nash Equilibrium.

Proof. This proof is considered under the bidder 1 perspective. Rewriting Equation 3.5 and using that $\max_{i\in N\setminus\{1\}}\{g(a_j) + h(c_i)\} = g(a_j) + h\left(\max_{i\in N\setminus\{1\}}\{c_i\}\right)$, because h is an increasing function, we can clear the term $g(a_j)$ in each case, and then, applying h^{-1} on both sides of both inequalities, we obtain the next expression:

$$\begin{aligned}
&\pi(\{x_j\}_{j\in M}, y) = \\
&\sum_{j\in M}\left[(a_j - g(x_j))P\left(\max_{i\in N\setminus\{1\}}\{c_i\} < h^{-1}(g(x_j) - g(a_j) + h(y))\right)\right] \\
&+ (c_1 - mh(y))P\left(\left\{\max_{i\in N\setminus\{1\}}\{c_i\} < h^{-1}(g(x_j) - g(a_j) + h(y))\right\}_{j\in M}\right).
\end{aligned} \tag{3.6}$$

We can rewrite Equation 3.6 as a piecewise function relating $\{a_j\}_{j\in M}$ with $\{x_j\}_{j\in M}$ on each piece. Particularly, we will work with the piece where $x_j = a_j$

for all $j \in M$ since that expression is reasonable in our search of a symmetric equilibrium. Thus, at that piece, the expected utility function is simplified as:

$$\pi(\{x_j\}_{j\in M}, y) = \left(\sum_{j\in M} [x_j - g(x_j)] + (c_1 - mh(y)) \right) P \left(\max_{i\in N\setminus\{1\}} \{c_i\} < y \right). \quad (3.7)$$

On the other hand, we have that $P\left(\max_{i\in N\setminus\{1\}} \{c_i\}\right) = F_C(y)$ where F_C is the cumulative distribution function of the maximum of extra profits. Because extra profit is a private and independent value we have that $F_C(x) = F(x)^{n-1}$, and then, the expected utility function is given by:

$$\pi(\{x_j\}_{j\in M}, y) = \left(\sum_{j\in M} [x_j - g(x_j)] + (c_1 - mh(y)) \right) F(y)^{n-1}. \quad (3.8)$$

We will denote $f'_x \equiv \frac{\partial f}{\partial x}$. Taking partial derivatives with respect to $\{x_j\}_{j\in M}$ and y from Equation 3.8 we have:

$$\pi'_{x_j} = \left[1 - g'_{x_j}(x_j)\right] F(y)^{n-1}; \quad (3.9)$$

$$\begin{aligned} \pi'_y =&(n-1)F(y)^{n-2} f(y) \left(\sum_{j\in M} [x_j - g(x_j)] + (c_1 - mh(y)) \right) \\ &- mh'_y(y)F(y)^{n-1}. \end{aligned} \quad (3.10)$$

We can take the first order conditions from 3.9 and 3.10 and, because of the symmetric equilibrium definition, we take $x_j = a_j$ for each item $j \in M$ and $y = c_1$. Now we are going to find the functional form of g and h such that the expected utility function π is maximized on those values. Thus, we obtain:

$$\left[1 - g'_{a_j}(a_j)\right] F(c_1)^{n-1} = 0 \quad (3.11)$$

$$\begin{aligned} &(n-1)F(c_1)^{n-2} f(c_1) \left(\sum_{j\in M} [a_j - g(a_j)] + c_1 - mh(c_1) \right) \\ &= mh'_{c_1}(c_1)F(c_1)^{n-1}. \end{aligned} \quad (3.12)$$

From Equation 3.11 we conclude that $g'_j(a_j) = 1$, and because of the initial condition $g(0) = 0$, we have $g(a_j) = a_j$. Substituting this in Equation 3.12 we obtain the next expression:

$$(n-1)F(c_1)^{n-2}f(c_1)(c_1 - mh(c_1)) = mh'_{c_1}(c_1)F(c_1)^{n-1}. \tag{3.13}$$

Notice that $[h(c_1)F(c_1)^{n-1}]'_{c_1} = [h'_{c_1}(c_1)F(c_1)^{n-1} + h(c_1)(n-1)F(c_1)^{n-2}f(c_1)]$ and grouping some terms we have the next condition:

$$m[h(c_1)F(c_1)^{n-1}]'_{c_1} = (n-1)c_1F(c_1)^{n-2}f(c_1). \tag{3.14}$$

Finally, using the fundamental theorem of calculus and isolating terms, we have:

$$h^*(c_1) = \frac{(n-1)\int_0^{c_1} yF(y)^{n-2}f(y)dy}{mF(c_1)^{n-1}}. \tag{3.15}$$

Now let us check that $(a_1 + h^*(c_1), a_2 + h^*(c_1), ..., a_m + h^*(c_1))$ is indeed an equilibrium. For doing that, we assume that each bidder $i \in N \setminus \{1\}$ bids according to $b_i^j = a_j + h^*(c_i)$ for all $j \in M$ and we will prove that bidding, in the same way, is the best response for bidder 1. Rewriting Equation 3.3 with this assumption we obtain:

$$\begin{aligned}\pi(\{b_1^j\}_{j\in M}) &= \sum_{j\in M}\left[\left(a_j - b_1^j\right)P\left(\max_{i\in N\setminus\{1\}}\{a_j + h^*(c_i)\} < b_1^j\right)\right] \\ &\quad + c_1P\left(\left\{\max_{i\in N\setminus\{1\}}\{a_j + h^*(c_i)\} < b_1^j\right\}_{j\in M}\right) \\ &= \sum_{j\in M}\left[\left(a_j - b_1^j\right)P\left(h^*\left(\max_{i\in N\setminus\{1\}}\{c_i\}\right) < b_1^j - a_j\right)\right] \\ &\quad + c_1P\left(\left\{h^*\left(\max_{i\in N\setminus\{1\}}\{c_i\}\right) < b_1^j - a_j\right\}_{j\in M}\right).\end{aligned} \tag{3.16}$$

Notice that if for all $j \in M$, $b_1^j \leq a_j$, the expected utility would be zero and, if that situation occurs for a subset of items, then the expected utility will be negative. So, for all $j \in M$, b_1^j must be greater than a_j to achieve a positive expected utility. Let $b_1^j = a_j + t_j$ be an expression for the bid. Rewriting 3.16 we have:

$$\begin{aligned}\pi(\{t_j\}_{j\in M}) &= \sum_{j\in M}\left[(-t_j)P\left(h^*\left(\max_{i\in N\setminus\{1\}}\{c_i\}\right) < t_j\right)\right] \\ &\quad + c_1P\left(\left\{h^*\left(\max_{i\in N\setminus\{1\}}\{c_i\}\right) < t_j\right\}_{j\in M}\right).\end{aligned} \tag{3.17}$$

Notice that when $t_j = t$, for all $j \in M$, bidder 1 only can get two possible outcomes which are to lose or to win the complete set of items, and also it represents the best response because bidder 1 always can choose t such that $mt \leq c_1$ for ensuring that his expected utility will not be negative. In addition, if we consider any other set $\{t_j\}_{j\in M}$ and that bidder 1 obtains a subset of goods, his expected utility would be negative; complimentarily, if all the goods are lost, his expected utility is equal to zero. Finally, if he wins all the goods, it is still the best response to take $t_k = t_l$, $\forall t, l$ to ensure not to overbid. Thus, considering $t_j = t$, $\forall j$, and

$$P\left(h\left(\max_{i\in N\setminus\{1\}}\{c_i\}\right) < t\right) = F_C(h^{-1}(t)) = F(h^*(t)^{-1})^{n-1}$$

we have:

$$\pi(t) = [c_1 - mt]F(h^*(t)^{-1})^{n-1}. \tag{3.18}$$

Using that $t_k = t_l$, $\forall t, l$ and considering symmetric equilibrium definition, which implies that all bidders rationalize their bids in the same way, we must have that $t = h(x)$ and we need to prove that $x = c_1$ maximizes the expected utility. Rewriting Equation 3.18 with these assumptions we have:

$$\pi(x) = [c_1 - mh(x)]F(x)^{n-1}. \tag{3.19}$$

Differentiating Equation 3.19 with respect to x, we obtain:

$$\pi'_x(x) = (n-1)F(x)^{n-2}f(x)[c_1 - mh(x)] - mh'_x(x)F(x)^{n-1}. \tag{3.20}$$

We have from Equation 3.13 the next equivalence:

$$h'_x(x) = \frac{(n-1)F(x)^{n-2}f(x)[x - mh(x)]}{mF(x)^{n-1}}. \tag{3.21}$$

Substituing Equation 3.21 into Equation 3.20, we obtain:

$$\begin{aligned}\pi'(x) =&(n-1)F(x)^{n-2}f(x)[c_1 - mh(x)] \\ &- \frac{mF(x)^{n-1}(n-1)F(x)^{n-2}f(x)[x - mh(x)]}{mF(x)^{n-1}} \\ &= (n-1)F(x)^{n-2}f(x)[c_1 - x].\end{aligned} \tag{3.22}$$

Then, if we consider that $x < c_1$ we have $\pi'(x) > 0$. Analogous reasoning can be made to show that if $x > c_1$ then $\pi'(x) < 0$ makes clear that $x = c_1$ maximizes the expected utility. □

Thereby, our equilibrium is given by $(a_1 + h^*(c_1), a_2 + h^*(c_1), ..., a_m + h^*(c_1))$ with h^* the well-known equilibrium bid for a first-price sealed-bid auction for a single good valued at c_1 split by m. Because h^* is an increasing function, the previous equilibrium ensures that the bidder with the highest valuation over the whole set of goods, wins all auctions, excluding the possibility that any bidder could obtain negative net profit by winning a subset of goods. In addition, our bidding function b satisfies the quasi-linear property in the equilibrium. That conclusion was unexpected since we only assume that $b(\cdot)$ is a linearly separable function.

Notice that if we integrate by parts we simplify $h^*(\cdot)$ as follows:

$$h^*(c_1) = \frac{(n-1)\int_0^{c_1} yF(y)^{n-2}f(y)dy}{mF(c_1)^{n-1}} = \frac{c_1}{m} - \frac{\int_0^{c_1} F(y)^{n-1}dy}{mF(c_1)^{n-1}} \tag{3.23}$$

To better understand the implication of our result we show the next example.

Example 3.1. *Consider a situation where bidders do not have any information about the valuation of the other bidders regarding to the extra profit, but they know that it can be any value in the interval* $[0,1]$*. Let us denote by* C *a random variable associated with the extra profit. So, we can assume that* C *has uniform distribution* $\mathcal{U}(0,1)$ *and for this particular example, we have that:*

$$h^*(c_1) = \frac{c_1}{m} - \frac{\int_0^{c_1} y^{n-1}dy}{mc_1^{n-1}} = \frac{(n-1)}{n}\frac{c_1}{m}. \tag{3.24}$$

So, for every $i \in N$ *and* $j \in M$*,* $b_i^j = a_j + \frac{(n-1)}{n}\frac{c_1}{m}$*.*

Expression 3.24 represents the extra amount that bidders are allowed to bid in each auction and implies that, in the equilibrium, the bids increase according to the number of bidders as well as decrease with the number of auctions and the player who has the highest extra profit valuation always gains the whole set of items.

To conclude this section we present a discussion case regarding more than one private independent value involved in the model, particularly the case of two auctions and two bidders where, both the individual valuations and the extra profit, are private and independent values. As we will show this case does not have a solution under our approach yet. For example, consider a situation where the individual valuations and the extra profit of each bidder are given according to the next table:

bidder 1	bidder 2
$a_1^1 = 0$	$a_2^1 = 0$
$a_1^2 = 30$	$a_2^2 = 12$
$c_1 = 12$	$c_2 = 26$

If we try to find a SBNE, assuming that bidders bid according to their valuation and a bidding function $b(a_i^j, c_i) = g(a_i^j) + h(c_i)$, we need to compare the bids to determine the allocation of the items:

	Item 1	Item 2
Bidder's 1 bid	$h(12)$	$g(30) + h(12)$
Bidder's 2 bid	$h(26)$	$g(12) + h(26)$

Now, if we assume that g and h are increasing functions, as we do in our model, the allocation of item 1 is clear because $h(26) > h(12)$ but the allocation of item 2 becomes a new problem because if item 2 is allocated to bidder 1, both bidders will have a negative utility as a result of the auctions. We will discuss this situation in more detail at the conclusions section.

3.2 Seller's Expected Revenue

In this Section, we analyze the seller's expected utility in our case of simultaneous auctions, and later we will compare it with the case where the complete set of goods is auctioned in a single first-price sealed-bid auction.

Theorem 3. *Let R^1 be the seller's expected profit in the multiple auctions for complementary goods case and $\hat{R}^1$ the seller's expected profit in a single first-price sealed-bid auction for the complete set of goods. Then, $R^1 = \hat{R}^1$.*

Proof. Let us denote $a \equiv \sum_{j \in M} a_j$. We assume that bidders will bid according to the equilibrium shown in the previous section, that is, $b_i^{j^*} = b^*(a_j, c_i) = g^*(a_j) + h^*(c_i) = a_j + h^*(c_i)$. Thus, the seller's expected revenue for the case of m simultaneous auctions is given by:

$$\begin{aligned} R^1 &= \mathbb{E}\left(a_1 + h^*\left(\max_{i \in N \setminus \{1\}} \{c_i\}\right) + \ldots + a_m + h^*\left(\max_{i \in N \setminus \{1\}} \{c_i\}\right)\right) \\ &= \mathbb{E}\left(a + mh^*\left(\max_{i \in N \setminus \{1\}} \{c_i\}\right)\right) = a + m \int_0^{\bar{c}} nh^*(c) F^{n-1}(c) f(c) dc. \end{aligned} \tag{3.25}$$

Consider the case where the seller decides to auction the complete set of goods by a single first-price sealed-bid auction. Also, assume that bidders will bid in equilibrium according to the valuation of the whole set of items, in the same way as they would for a single item with a valuation given by $v_i = a + c_i$.

To determine the equilibrium bid for this case, we need to consider that $\forall i \in N$, v_i is a private and independent value in $[a, a + c_i]$. Analyzing the game from the point of view of bidder 1, modeling it as a Bayesian game, and assuming that $\{v_i\}_{i \in N}$ are independent realizations from an identical random variable V with cumulative distribution $F_V(\cdot)$ and density $f_V(\cdot) > 0$ in the interval $[a, a + \bar{c}]$, we can write the next expression:

$$\pi(b_1) = (v_1 - b_1) P\left(\max_{i \in N \setminus \{1\}} \{b_i\} < b_1\right). \tag{3.26}$$

Assuming that all players follow a bidding strategy $\hat{b}(\cdot)$ and bidders $i \in N \setminus \{1\}$ bid according to their value we have that $b_i = \hat{b}(v_i)$ and we obtain the next equivalence.

$$\begin{aligned}\pi(\hat{b}(x)) &= (v_1 - \hat{b}(x))P\left(\max_{i\in N\setminus\{1\}}\left\{\hat{b}(v_i)\right\} < \hat{b}(x)\right)\\ &= (v_1 - \hat{b}(x))P\left(\hat{b}\left(\max_{i\in N\setminus\{1\}}\{v_i\}\right) < \hat{b}(x)\right)\\ &= (v_1 - \hat{b}(x))P\left(\max_{i\in N\setminus\{1\}}\{v_i\} < x\right).\end{aligned} \tag{3.27}$$

If we define $\hat{V} \equiv \max_{i\in N\setminus\{1\}}\{v_i\}$ as the random variable associated with the maximum value and denote by $F_{\hat{V}}(\cdot)$ as the cumulative distribution function of $\hat{V}$. Then we have that $P\left(\max_{i\in N\setminus\{1\}}\{v_i\} < x\right) = F_{\hat{V}}(x)$ and because $\forall i \in N$ value v_i is private and independent, we have that $F_{\hat{V}}(x) = F_V(x)^{n-1}$. Thus, the expected utility function is given by the following expression:

$$\pi(\hat{b}(x)) = (v_1 - \hat{b}(x))F_V(x)^{n-1}. \tag{3.28}$$

From 3.28 and following the classical approach for single items first-price sealed-bid auctions we obtain the next bidding equilibrium:

$$\hat{b}^*(v_1) = \frac{(n-1)\int_a^{v_1} xF_V(x)^{n-2}f_V(x)dx}{F_V(v_1)^{n-1}}. \tag{3.29}$$

Denoting by C the random variable in $[0,\bar{c}]$ with distribution $F(\cdot)$ and density $f(\cdot)$ we can write $V = a + C$ and then we have $F_V(x) = P(V < x) = P(a+C < x) = P(C < x-a) = F(x-a)$. Thus, we have the next equivalence:

$$\hat{b}^*(v_1) = \frac{(n-1)\int_a^{v_1} xF(x-a)^{n-2}f(x-a)dx}{F(v_1-a)^{n-1}}. \tag{3.30}$$

If we take $x - a = y$ we obtain:

$$\hat{b}^*(v_1) = \frac{(n-1)\int_0^{v_1-a}(y+a)F(y)^{n-2}f(y)dy}{F(v_1-a)^{n-1}}. \tag{3.31}$$

Moreover, since for all $i \in N$ we have $v_i = a+c_i$ we have the next espression:

$$\begin{aligned}\hat{b}^*(a+c_1) =& \frac{(n-1)\int_0^{c_1}(y+a)F(y)^{n-2}f(y)dy}{F(c_1)^{n-1}}\\ =& \frac{(n-1)\int_0^{c_1} yF(y)^{n-2}f(y)dy}{F(c_1)^{n-1}} + a = \hat{b}^{**}(c_1) + a\end{aligned} \tag{3.32}$$

where b^{**} is the well-known equilibrium bid for a first-price sealed-bid auction. Thus, we have that the seller's expected revenue, in this case, is given by:

$$\begin{aligned}\hat{R}^1 &= \mathbb{E}\left(a + \max_{i\in N\setminus\{1\}}\left\{\hat{b}^{**}\left(c_i\right)\right\}\right) = a + \mathbb{E}\left(\hat{b}^{**}\left(\max_{i\in N\setminus\{1\}}\{c_i\}\right)\right) \\ &= a + n\int_0^{\bar{c}} \hat{b}^{**}(c)F^{n-1}(c)f(c)dc.\end{aligned} \tag{3.33}$$

Finally, if we notice that $mh^*(c) = \hat{b}^{**}(c)$ the result follows immediately. □

Although the construction of our model is different, the previous result is not surprising because the SBNE expression validates prior studies, and the methodology to obtain it is the classical one.

On the other hand, if we analyze Example 3.1 from the seller's expected revenue perspective, taking into account that $\hat{b}^{**}(c) = \frac{(n-1)}{n}c$, we have that:

$$R^1 = a + (n-1)\int_0^1 c^n dc = a + \frac{n-1}{n+1}. \tag{3.34}$$

Thus, it is easy to verify that the seller's expected revenue does not depend on the number of auctions but it is an increasing function of the number of players. The fact splitting the auctions individually does not influence the seller's expected revenue was, at first, unintuitive as well as the result of Theorem 2

With respect to the cases discussed in the previous section involving more than one private independent value, the seller's revenue perspective could be a clue for solving that generalization if the search for the solution focuses on offering an allocation that maximizes the seller's expected revenue.

3.3 Conclusions

The previous approach offers a solution from the perspective of strategic behavior for the simultaneous auctions problem where complementary goods play an important role. Although the assumptions for obtaining such a solution are quite desirable, the problem still has some interesting open questions.

One case was shown in Example 3.1, which is a direct generalization of the problem we solve when we consider more than one private independent value. Under our model, it is not clear how to compare bids. If we propose a way how to do it, we could provide a solution for the allocation problem.

It is clear that we have to make new assumptions over the bidding function in order to solve the case from the strategic behavior perspective, even to propose a completely different bidding function that guarantees efficient assignations implying that no bidder obtains negative profit to ensure that bidders will be interested in being part of the game. On the other hand, we have the seller's point of view, where is interesting to propose an allocation that maximizes the seller's expected revenue.

Another possible generalization occurs when there is a case where for each subset of goods there is a private independent extra profit c_i^J, $J \subset M$. For this situation, the main problem lies in the relation amongst several random variables in the utility functions and we have to handle the fact that it is not trivial that the same assumptions work as well as in the previous model. We can consider another open problem with similar difficulties where the extra profit value is common knowledge but the individual valuations are independent private values.

For solving any generalized case, we know that different assumptions need to be implemented and we have to focus on what kind of allocation will be induced by that assumptions. The results of this chapter have been published in *Dynamics and Games Journal*.

Chapter 4

Simultaneous auctions of identical items by first-price close mechanism with a bonus for buyers.

In Chapter 3 we propose a solution for a situation where m different items were sealed by simultaneous auctions mechanisms. We considered that each item's individual valuation was common knowledge and the only private and independent value involved was the extra profit value. Nevertheless, Section 3.3 discusses many interesting and unsolved situations for more than one private value. The main purpose of this chapter is to offer a solution for a case where items' individual valuation and extra profit (bonus for buyers) are private independent information. Thus, this chapter focuses on studying from the seller's perspective, how to auction a finite set of m identical items when the buyer who obtains the whole set of goods (if any) gets a bonus, such that the seller maximizes his expected revenue taking into account the strategic behavior of the bidders.

4.1 The model.

In order to be consistent with the notation proposed in Chapter 3, and considering that players face a situation of m simultaneous auctions for identical items, we say that a_i is one item individual valuation for player $i \in N$. On the other hand, we have that c_i is player i's extra profit value. Let us define an increasing function $H : [0, \bar{a}] \rightarrow \mathbb{R}_+$ which is common knowledge for bidders. This function represents the rule that determines the value of the extra profit, namely, $c_i \equiv H(a_i)$. To come clean with this, consider that, $\forall i \in N$, $H_c(a_i) = c$, this means that if some player wins the whole set of items he wins an extra amount c. Now, consider $H_k(a_i) = ka_i$, with $k \in \mathbb{N}$, which means that if a player wins the whole set, the extra profit consists of k extra items. In any case, we interpret $H(\cdot)$ as a bonus that the seller offers to encourage bidders to go for the whole set, in common words, that could be the case in which a seller offers free shipping or some kind of reward if a bidder wins all simultaneous auctions. Thus, we have that $v_i = ma_i + H(a_i)$ for this case. Since we did not change Section's 3.1 mechanism allocation rules, following the same methodology to find a $SBNE$ and supposing in addition that bidders bid the same amount for items with the same valuations, we propose that for every $i \in N$ and $j \in M$ we have $b_i^j = b_i$. Thus, we can write Equation 3.3 as follows:

$$\pi(b_1) = \sum_{j \in M} \left[(a_1 - b_1) P \left(\max_{i \in N \setminus \{1\}} \{b_i\} < b_1 \right) \right] \\ + H(a_1) P \left(\left\{ \max_{i \in N \setminus \{1\}} \{b_i\} < b_1 \right\}_{j \in M} \right). \quad (4.1)$$

Notice that since index j does not have a direct influence in either the terms of Equation 4.1 and then, we can write the next equation:

$$\pi(b_1) = [m(a_1 - b_1) + H(a_1)] P \left(\max_{i \in N \setminus \{1\}} \{b_i\} < b_1 \right). \quad (4.2)$$

Keeping the assumption that bidders bid according to an increasing bivariate function $b(\cdot)$ and according to the same criteria as in the previous chapter, namely $b_i = b(a_i, H(a_i))$ we define $\alpha(a_i) \equiv b(a_i, H(a_i))$ as a non negative increasing function that satisfy $\alpha(0) = 0$. Thus, we have the next expression:

$$\pi(x) = [m(a_1 - \alpha(x)) + H(a_1)] P \left(\max_{i \in N \setminus \{1\}} \{\alpha(a_i)\} < \alpha(x) \right). \quad (4.3)$$

Given Equation 4.3 we can offer a $SBNE$ determining the best response for bidder 1 given that the other bidders bid according to $\alpha(a_i)$. Thus, our first result is given by the following Theorem:

Theorem 4. *For the case of m first-price seald-bid simultaneous auctions and n bidders, where the auctioned goods are considered to be identical, the* *individual valuation is a private and independent value, the extra profit is defined by $H(\cdot)$ and the bid strategy is given by $\alpha(\cdot)$,*

$$b_i = b(a_i, H(a_i)) = \alpha^*(a_i)$$

where:

$$\alpha^*(a_i) = \frac{(n-1)\int_0^{a_i} xF(x)^{n-2}f(x)dx}{F(a_i)^{n-1}} + \frac{(n-1)\int_0^{a_i} H(x)F(x)^{n-2}f(x)dx}{mF(a_i)^{n-1}}$$
$$= \frac{(n-1)\int_0^{a_i}(x+\frac{H(x)}{m})F(x)^{n-2}f(x)dx}{F(a_i)^{n-1}};$$

is a SBNE.

Proof. Notice that, since $\alpha(\cdot)$ is an increasing function we have

$$P\left(\max_{i\in N\setminus\{1\}}\{\alpha(a_i)\} < \alpha(x)\right) = P\left(\alpha\left(\max_{i\in N\setminus\{1\}}\{a_i\}\right) < \alpha(x)\right).$$

Then, taking α^{-1} in both sides of the inequality, we obtain:

$$\pi(x) = [m(a_1-\alpha(x)) + H(a_1)]P\left(\max_{i\in N\setminus\{1\}}\{a_i\} < x\right). \quad (4.4)$$

Let us denote by $A \equiv \max_{i\in N\setminus\{1\}}\{a_i\}$ the random variable associated to the maximum of valuations with distribution $F_A(\cdot)$. Considering independence between individual valuations we can write that

$$P\left(\max_{i\in N\setminus\{1\}}\{a_i\} < x\right) = F_A(x) = F(x)^{n-1},$$

where $F(\cdot)$ is the distribution of the individual values. Then, we have:

$$\pi(x) = [m(a_1-\alpha(x)) + H(a_1)]F(x)^{n-1}. \quad (4.5)$$

Taking derivative of π with respect to x, we obtain the next expression:

$$\pi'(x) = (n-1)F(x)^{n-2}f(x)[m(a_1-\alpha(x)) + H(a_1)] - m\alpha'(x)F(x)^{n-1}. \quad (4.6)$$

Taking the first order conditions and making $x = a_1$ for the symmetrical equilibrium definition we have:

$$(n-1)F(a_1)^{n-2}f(a_1)[m(a_1-\alpha(a_1)) + H(a_1)] = m\alpha'(a_1)F(a_1)^{n-1}. \quad (4.7)$$

Rearranging some terms we obtain the following equation:

$$m[\alpha(a_1)F(a_1)^{n-1}]' = [ma_1 + H(a_1)](n-1)F(a_1)^{n-2}f(a_1). \tag{4.8}$$

Then, by applying the Fundamental Theorem of Calculus we have:

$$m\alpha(a_1)F(a_1)^{n-1} = (n-1)\int_0^{a_1}[(mx + H(x)]F(x)^{n-2}f(x)dx. \tag{4.9}$$

Clearing and separating the integrals on the right, we obtain the following expression:

$$\alpha^*(a_1) = \frac{(n-1)\int_0^{a_1} xF(x)^{n-2}f(x)dx}{F(a_1)^{n-1}} + \frac{(n-1)\int_0^{a_1} H(x)F(x)^{n-2}f(x)dx}{mF(a_1)^{n-1}}. \tag{4.10}$$

Equation 4.10 gives the functional form of the bid given by $\alpha^*(\cdot)$. In order to prove that the expression is indeed a $SBNE$, we must prove that since the bidders $i \in N \setminus \{1\}$ bid according to $\alpha^*(a_i)$, bidder's 1 expected utility is maximized when $x = a_1$. To achieve this purpose we will analyze the sign change in $\pi'(\cdot)$ when $x \neq c_1$.

Let be $x < c_1$. From Equation 4.7 we have that

$$\alpha'(a_1) = \frac{(n-1)F(a_1)^{n-2}f(a_1)[m(a_1 - \alpha(a_1)) + H(a_1)]}{mF(a_1)^{n-1}}.$$

Using this fact and evaluating π at x we get:

$$\begin{aligned}\pi'(x) =& (n-1)F(x)^{n-2}f(x)[m(a_1 - \alpha(x)) + H(a_1)] \\ &- \frac{mF(x)^{n-1}(n-1)F(x)^{n-2}f(x)[m(x - \alpha(x)) + H(x)]}{mF(x)^{n-1}}.\end{aligned} \tag{4.11}$$

Simplifying Equation 4.11 we obtain:

$$\pi'(x) = (n-1)F(x)^{n-2}f(x)[m(a_1 - x)] > 0 \tag{4.12}$$

With similar reasoning, we can prove that if $x > c_1$, then $\pi'(x) < 0$. □

It is easy to note that

$$\alpha^*(a_i) = \frac{(n-1)\int_0^{a_i} xF(x)^{n-2}f(x)dx}{F(a_i)^{n-1}} + \frac{(n-1)\int_0^{a_i} H(x)F(x)^{n-2}f(x)dx}{mF(a_i)^{n-1}}$$

is an increasing function on a_i. Thus, we conclude $\alpha^*(\cdot)$ is a $SBNE$ that ensures that whoever values the set of goods the most will win all auctions, excluding the possibility that some bidders could obtain negative utility by obtaining only a subset of items.

On the other hand, we observe that the equilibrium bid $\alpha^*(\cdot)$ is composed by two terms, where the term on the left corresponds to the well-known expression of the equilibrium bid of the first-price sealed-bid auction for an item valued in a_i. Further, the term on the right is interpreted as the additional amount that bidders are willing to bid in each auction given that sellers offer $H(\cdot)$. For example, if we consider a case where the seller offers $H_c(x)$, we have that $\alpha^*(\cdot)$ has the next value:

$$\frac{(n-1)\int_0^{a_i} H_c(x)F(x)^{n-2}f(x)dx}{mF(a_i)^{n-1}} = \frac{c\int_0^{a_i}(n-1)F(x)^{n-2}f(x)dx}{mF(a_i)^{n-1}} = \frac{c}{m}. \quad (4.13)$$

The foregoing shows that the additional amount that bidders bid for each good is equivalent to distributing the value of the bonus in m equal parts. Now, if we consider $H_k(x)$ instead of $H_c(x)$, we have:

$$\frac{(n-1)\int_0^{a_i} H_k(x)F(x)^{n-2}f(x)dx}{mF(a_i)^{n-1}} = \frac{k}{m}\frac{(n-1)\int_0^{a_i} xF(x)^{n-2}f(x)dx}{F(a_i)^{n-1}}. \quad (4.14)$$

This shows that the additional amount that the bidders' bid in this case is equivalent to distributing in $\frac{k}{m}$ equal parts the well-known first-price sealed bid equilibrium for an item valued in a_i.

In the next section, we will carry out the analysis corresponding to the seller's expected revenue, where we consider the possible changes in his utility considering different values of $H(\cdot)$. Likewise, we will analyze the expected utility of the seller when he implements a single auction under the closed mechanism of the first price and the only opportunity, to auction the complete set of goods together with the extra utility.

4.1.1 Seller's Expected Revenue.

In this section, we will analyze the seller's expected revenue for our case of simultaneous auctions. Our analysis focuses on studying different proposals regarding the function $H(\cdot)$, which describes the way in which the bonus for bidders is defined.

It is assumed that the seller has a number of goods denoted by $r \in \mathbb{N}$. The seller has to figure out how many items to auction and how many to give as a bonus in order to obtain a better expected revenue. For example, consider $H_k(\cdot)$ then $r = m + k$, and if we consider $H_c(\cdot)$ we have that $r = m$ and the seller will offer a fixed amount as a bonus. To synthesize these two particularly useful functional forms we define $H_{k,c}(x) \equiv kx + c$ and analyze the seller's expected revenue under these conditions.

Denoting by R^1 the seller's expected revenue for the case of simultaneous auctions and assuming that bidders follow the bidding strategy $\alpha^*(\cdot)$ we obtain the next expression:

$$R^1 = \mathbb{E}\left(m\alpha^*\left(\max_{i\in N}\{a_i\}\right)\right) = m\mathbb{E}\left(\alpha^*\left(\max_{i\in N}\{a_i\}\right)\right). \tag{4.15}$$

Since good individual valuation is given by an independent private value, we have that $F_A(x) = F(x)^n$ and therefore $f_A(x) = nF(x)^{n-1}f(x)$. Thus, the seller's expected profit is given by:

$$R^1 = m\int_0^{\bar{a}} \alpha^*(x)nF(x)^{n-1}f(x)dx. \tag{4.16}$$

Considering $H_{k,c}(\cdot)$ and that the seller owns r identical and indivisible goods and a fixed amount $c > 0$ to give as an incentive, we can analyze the way that the seller distributes r. Rewriting 4.16 under these assumptions we obtain the following expression:

$$R^1 = m\int_0^{\bar{a}} \left[\frac{(n-1)\int_0^x (y + \frac{ky}{m} + \frac{c}{m})F(y)^{n-2}f(y)dy}{F(x)^{n-1}}\right] nF(x)^{n-1}f(x)dx. \tag{4.17}$$

Rearranging some terms we obtain the following equivalence:

$$\begin{aligned} R^1 =& m\int_0^{\bar{a}} \left[\frac{(n-1)\int_0^x (1 + \frac{k}{m})yF(y)^{n-2}f(y)dy}{F(x)^{n-1}}\right] nF(x)^{n-1}f(x)dx. \\ &+ m\int_0^{\bar{a}} \left[\frac{(n-1)\int_0^x \frac{c}{m}F(y)^{n-2}f(y)dy}{F(x)^{n-1}}\right] nF(x)^{n-1}f(x)dx. \end{aligned} \tag{4.18}$$

Let us note that:

$$\frac{(n-1)\int_0^x \frac{c}{m}F(y)^{n-2}f(y)dy}{F(x)^{n-1}} = \frac{\frac{c}{m}\int_0^x (n-1)F(y)^{n-2}f(y)dy}{F(x)^{n-1}} = \frac{c}{m}. \tag{4.19}$$

Similarly, we see that:

$$m\int_0^{\bar{a}} \frac{c}{m}nF(x)^{n-1}f(x)dx = cF(\bar{a}) = c \tag{4.20}$$

Let us denote by $\hat{b}^*(x) \equiv \frac{(n-1)\int_0^x yF(y)^{n-2}f(y)dy}{F(x)^{n-1}}$ the well-known first-price sealed-bid auction equilibrium for a good valued at x. Substituting Equation 4.20 into Equation 4.18 and factorising the term $(1 + \frac{k}{m})$ we get:

$$
\begin{aligned}
R^1 &= (m+k)\int_0^{\bar{a}} \hat{b}^*(x) n F(x)^{n-1} f(x) dx + c \\
&= r\int_0^{\bar{a}} \hat{b}^*(x) n F(x)^{n-1} f(x) dx + c.
\end{aligned} \tag{4.21}
$$

The previous expression shows that no matter how the seller distributes the r goods, his expected revenue is always given by the same amount. Also, we observe that when part of the incentive offered by the seller is given by a constant $c > 0$, it happens that the seller always recovers exactly the said value, so it does not have a decisive influence on his expected income either.

We will now determine the seller's expected revenue when he decides to auction the complete set of items along with the bonus through a single, first-price sealed-bid auction. Suppose that bidders $i \in N$ bid in equilibrium according to the valuation of the complete set of goods, in the same way as if a single item whose valuation for bidder i was given by $v_i = ma_i + H(a_i)$ were auctioned.

We will model the above situation as a Bayesian Game considering the case where the extra utility is given by $H_{k,c}(\cdot)$. Let $\hat{A}$ be the random variable in the interval $[0, \bar{a}]$ with distribution function $F(\cdot)$ and density $f(\cdot)$ and let $V \equiv r\hat{A}+c$ be the random variable in the interval $[c, r\bar{a}+c]$ with distribution function $F_V(\cdot)$ and density $f_V(\cdot)$. Thus, we have that $v_i = ra_i + c$ is a private and independent value that represents the valuation of the bidder i for the complete set of goods. Then, the general expression of expected utility for the bidder 1 is given by:

$$
\pi(b_1) = (v_1 - b_1) P\left(\max_{i \in N\setminus\{1\}} \{b_i\} < b_1\right), \tag{4.22}
$$

where b_i represents player i's bid. Following the classical methodology considering that the players bid according to an increasing and non-negative function $\hat{b}(\cdot)$ such that $\hat{b}(0) = 0$ we have:

$$
\begin{aligned}
\pi(\hat{b}(y)) &= (v_1 - \hat{b}(y)) P\left(\max_{i \in N\setminus\{1\}} \left\{\hat{b}(v_i)\right\} < \hat{b}(y)\right) \\
&= (v_1 - \hat{b}(y)) P\left(\hat{b}\left(\max_{i \in N\setminus\{1\}} \{v_i\}\right) < \hat{b}(y)\right) \\
&= (v_1 - \hat{b}(y)) P\left(\max_{i \in N\setminus\{1\}} \{v_i\} < y\right).
\end{aligned} \tag{4.23}
$$

Note that $P\left(\max_{i \in N\setminus\{1\}} \{v_i\} < y\right) = F_V(y)^{n-1}$ and thus we obtain the following expression for the expected utility:

$$
\pi(y) = (v_1 - \hat{b}(y)) F_V(y)^{n-1}. \tag{4.24}
$$

We obtain the equilibrium bid from Equation 4.24 as follows:

$$\hat{b}(v_1) = \frac{(n-1)\int_c^{v_1} yF_V(y)^{n-2}f_V(y)dy}{F_V(v_1)^{n-1}}. \tag{4.25}$$

Using that $F_V(y) = P(V < y) = P(r\hat{A} + c < y) = P(\hat{A} < \frac{y-c}{r}) = F(\frac{y-c}{r})$ and substituting this in Equation 4.25 we get the following equivalence:

$$\hat{b}(v_1) = \frac{(n-1)\int_c^{v_1} yF(\frac{y-c}{r})^{n-2}f(\frac{y-c}{r})dy}{F(\frac{v_1-c}{r})^{n-1}}. \tag{4.26}$$

Now, making $x = \frac{y-c}{r}$ we obtain:

$$\hat{b}(v_1) = \frac{(n-1)\int_0^{\frac{v_1-c}{r}} (rx+c)F(x)^{n-2}f(x)dx}{F(\frac{v_1-c}{r})^{n-1}}. \tag{4.27}$$

Furthermore, if we substitute that $v_1 = ra_1 + c$, we have:

$$\hat{b}(ra_1 + c) = \frac{(n-1)\int_0^{a_1} (rx+c)F(x)^{n-2}f(x)dx}{F(a_1)^{n-1}}. \tag{4.28}$$

And rearranging 4.28 we obtain:

$$\hat{b}(ra_1 + c) = \frac{r(n-1)\int_0^{a_1} xF(x)^{n-2}f(x)dx}{F(a_1)^{n-1}} + \frac{c\int_0^{a_1} (n-1)F(x)^{n-2}f(x)dx}{F(a_1)^{n-1}}. \tag{4.29}$$

Finally solving the integral on the right and substituting $\hat{b}^*(x)$ properly, we obtain:

$$\hat{b}(ra_1 + c) = r\hat{b}^*(a_1) + c. \tag{4.30}$$

Thus, Equation 4.30 shows that the seller does not care to implement any of the two mechanisms since he always obtains the same expected revenue.

4.2 Conclusions.

For concluding this chapter, we want to mention that, from a theoretical perspective, bidders act according to two situations where they choose their strategies in two different spaces: one for a simultaneous auctions context; and two for a single auction context. Considering the previous observation it was not very intuitive that we will obtain a result that implies that the way to bid in both cases was so similar even less considering that the bonus was given by $H_{k,c}(\cdot)$.

A direct generalization of this model is to consider more than one private and independent value, for example, consider two simultaneous auctions for m different goods where each bidder has a personal valuation for each of them and likewise consider the bonus as a private independent value. Another important generalization could consider that there was more than one value related to the extra profit, for example, if m auctions are considered, there could be an extra profit value associated with each of the 2^m subsets of goods. Latter and former cases remain open questions even if they preserve the first-price mechanism or another, even whether the goods are identical or different.

On the other hand, we like to point out that if we try to compare both situations from an empirical perspective, there are several reasons to suppose that bidders should not act in such similar ways in both situations. One of the reasons is that in the case of m simultaneous auctions, it is possible that some bidder could obtain negative utility if he bids above the valuation of the object and he only obtained a subset of goods. Another reason could be due to the fact that in the case of m simultaneous auctions, the way of acting of the other bidders could be somewhat less predictable since it would be necessary to analyze the way in which they could rationalize m possibly different bids for each auction. This type of analysis is really interesting to approach from the perspective of the experimental economics area in order to explain for what kind of applications $SBNE$ will be a proper tool to predict agents' behaviour. The results of this chapter have been published in *EconoQuantum* journal.

In the next chapter, we explore the information exchange for a supply chain context modeling it by a signaling game. Later we explore the application of auctions as a negotiating mechanism among agents involved in the supply chain.

Chapter 5

Applications of signal games and auctions to supply chains.

This chapter explores a different approach to supply chain modeling through the game theory perspective. In Section 5.1, we explain a two-echelon one single channel supply chain under common assumptions through a signaling game with the classical structure. We prove that, in our first approach, there are many different ways to obtain an equilibrium but most of these strategies do not necessarily imply cooperative behavior between agents. Later, we propose some modifications to the original assumptions which let us obtain a unique equilibrium that ensures cooperative behavior among agents.

In Section 5.2 we analyze a situation where only one supplier and several buyers are involved in a negotiation process when the supplier is not capable of meeting the sum of the buyers' individual demands. We propose for this case to assign the production lot through an auction mechanism where buyers state the number of goods they want and the unit price they are willing to pay for each item in the case they win. We consider that bidders have only one chance to make a sealed bid. Later, the supplier orders the bids from highest to lowest giving preference to the highest bid to meet that demand, then to the second highest bid, and so on until he finishes allocating the full amount of goods. For their part, buyers pay the number of goods that were assigned to them at a unit price given by their bid. This mechanism has a canonic case which implies that all bidders bid zero as the unit price. For that case, the model considers that if some bidder bids under the market price $\hat{p}$ then he has to pay that price for each unit.

5.1 Signaling game for modeling a supply chain.

A signaling game defines a situation where one or more informed agents, commonly called senders, take observable decisions before one or more uninformed agents, commonly called receivers. A signaling game is performed as follows:

- Nature determines informed agents' types.
- Informed agents decide their actions depending on their types.
- Uninformed agents observe informed agents' actions and choose their own actions.

As we said before, the fundamental characteristic of a signaling game lies in the fact that agents are asymmetrically informed. Thus, the uninformed agents have to wait for the informed ones to act. This means that the uninformed agents wait for a signal before choosing an action.

Particularly, we develop a situation with only one supplier and a retailer that negotiates the quantity of a particular item to be produced and commercialized, respectively. Our model identifies nature as the market demand and supposes that, since the retailer is closer to the market and therefore has better information about it, he is the informed agent, that is to say, the sender, and thus the supplier will be the receiver. Also, to keep it simple, we assume that the transmitter will observe the real market demand and will be able to send one of two possible signals: *low* demand or *high* demand.

Now, we will explain the variables involved in our decision-making supply chain problem. Later, we will analyze the situation as a signaling game to find out the characteristics of the information exchange about the market demand. For simplicity, we will consider that the costs associated with the different variables are unit costs.

We will denote by c_R the commercialization unit cost for the retailer and by c_S the unit cost of production for the supplier. In this model, we assume that the retailer's commitment regarding the merchandise to be sold is always for the full quantity that the supplier decides to manufacture, whether the supplier follows the signal or not. For example, if the supplier receives a signal that the demand will be low and still decides to produce a large batch, the retailer commits to market the entire batch. Thus, the total commercialization cost for the retailer will be given according to the number of manufactured goods.

We will denote by so_R and so_S the unit costs that the retailer and supplier assume respectively due to stock-outs, that is, the loss for having produced or commercialized fewer units than the market requires. In this model, we assume that this cost always affects both agents.

In the same way, let h be the unit cost due to the unsold units, that is, the opposite case of stock-outs since a fixed amount per unit is paid due to the cost generated by removing the surplus items from the market. The return policies for this case imply that since the supplier is in charge of transport logistics and who decides how to distribute his merchandise, in this model it is assumed that this cost will always be absorbed by his side.

On the other hand, p_r represents the unitary market price at which the retailer sells the merchandise and r is the unitary price that the retailer pays to the supplier for each unit of product. Here, it is assumed that the purchase commitment policies imply that the retailer only commits to pay for the units that he has managed to retail. For simplicity, we assume that the supplier has to decide between producing q_+ or q_- a big or a small lot of product respectively, and the retailer only can send q_+ or q_- as the market status signals namely big or small demand respectively. The random variable associated with the market demand will then have a Bernoulli distribution with parameter $p_+ \in [0, 1]$ and it will be denoted by $N \sim \mathcal{B}(p_+)$. Furthermore our approach, it is known that the retailer knows the true market demand and that the supplier only knows the probabilistic distribution. The quantity produced will be denoted by Q.

Formally, the signaling game strategies will be the following:

- The retailer will decide which signal to send to the supplier depending on the observed demand, that is, $S_R = \{(q_+, q_-), (q_-, q_+), (q_-, q_-), (q_+, q_+)\}$.
- The supplier will decide how much to produce in each information set, that is, $S_S = \{(q_+, q_-), (q_-, q_+), (q_-, q_-), (q_+, q_+)\}$.

The expected utility of the agents will be given by:

$$\pi_R = \mathbb{E}[(p_r - r - c_R)\min\{Q, N\} - so_R(N - Q)^+]. \tag{5.1}$$

$$\pi_S = \mathbb{E}[r\min\{Q, N\} - c_S Q - so_S(N - Q)^+ - h_S(Q - N)^+], \tag{5.2}$$

where π_R and π_S represent the expected return of the retailer and the supplier respectively, considering $(x)^+ \equiv \max\{0, x\}$.

Our signaling game performance consists of three stages. First: Nature reveals market demand to the retailer. Second: the retailer sends a signal, *low* or *high*, to the supplier. Third: the supplier will determine the production level based on the signal he received. Let α denote the supplier's belief of having received the signal q_- when the true market demand is q_+, and β the belief associated with receiving the signal q_+ when this is the true market demand. Thus, we can represent the situation through the next signaling game:

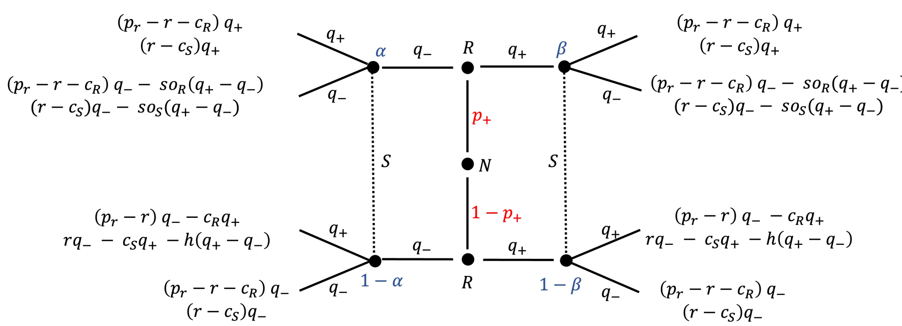

Figure 5.1: Signaling game G_1.

With the previous construction, we can state our first Theorem:

Theorem 5. *For the $G1$ signaling game, the strategies:*

$$s_{G1}^{s1} = ((q_+, q_-), (q_-, q_+))$$
$$s_{G1}^{s2} = ((q_-, q_+), (q_+, q_-))$$

are separating equilibria. Moreover, if the next inequality holds

$$(p_+)[(r - c_s)q_+] + (1 - p_+)[rq_- - c_s q_+ - h(q_+ - q_-)] \\ > (p_+)[(r - c_s)q_- - so_S(q_+ - q_-)] + (1 - p_+)[(r - c_S)q_-] \quad (5.3)$$

we have that for all $a \in [0, 1]$,

$$s_{G1}^{p1} = ((q_-, q_-), (q_+, (aq_+, (1 - a)q_-)))$$
$$s_{G1}^{p3} = ((q_+, q_+), (q_+, (aq_+, (1 - a)q_-)))$$

are pooling equilibria. Finally, if 5.3 does not hold then,

$$s_{G1}^{p2} = ((q_-, q_-), (q_-, q_-))$$
$$s_{G1}^{p2} = ((q_+, q_+), (q_-, q_-))$$

are pooling equilibria.

Proof. This proof will be divided into two parts according to the signal type.

Separating signals: In this part, we will analyze two strategies that imply that, when the sender (retailer) sends the signal, the receiver (supplier) must decide his best response considering that certainty he does not know in which information set he is. That is, at the moment that he receives the signal, he must determine the best response for each information set.

The strategy $s_R = (q_+, q_-)$ is interpreted as the retailer always yielding up the true market signal to the supplier, ie, he cooperates with him. The orange lines in the following graph represent the circumstance under which we analyze the agents' decisions.

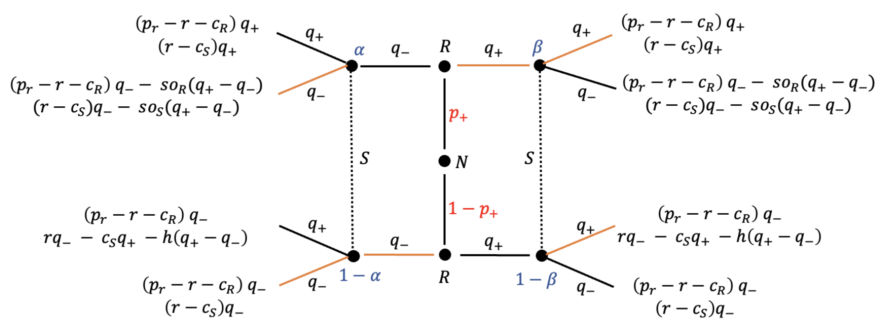

Figure 5.2: G_1 under *true* separating signals.

For this case we have that $\alpha = 0$ and $\beta = 1$, which implies that S will choose q_- on the information set of α because $(r - c_S)q_- > rq_- - c_S q_+ - h(q_+ - q_-)$ always happens; and S will choose q_+ on the information set of β whenever $(r - c_S)q_+ > (r - c_S)q_- - so_S(q_+ - q_-)$, which always holds, too. Likewise, we observe that if R changes its strategy in the node associated with p_+ he does not get improvement. A similar happens in the node associated with $(1 - p_+)$. Therefore the strategy $s^{s1}_{G1} = ((q_+, q_-), (q_-, q_+))$ is a separating equilibrium of G_1.

The analysis of the strategy $s_R = (q_-, q_+)$ is analogous but with the opposite interpretation. This strategy implies that the retailer always sends the wrong signal to the supplier, that is, he always lies to him. The extensive form of the game for this case is given below:

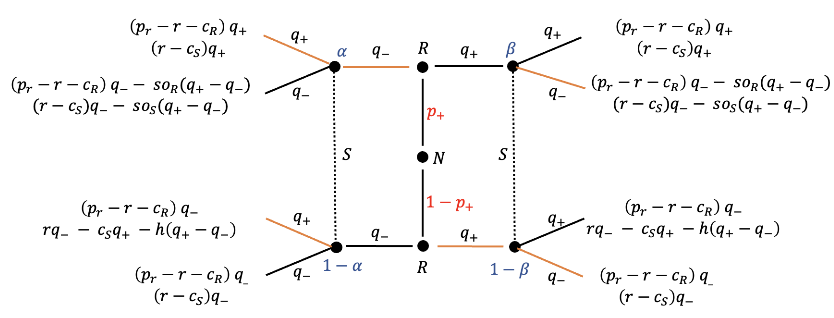

Figure 5.3: G_1 under *false* separating signals.

For this case, we have that $\alpha = 1$ and $\beta = 0$. Thus, S must decide q_+ in the information set of α because $(r - c_S)q_+ > (r - c_S)q_- - so_S(q_+ - q_-)$ is always true; and he will decide q_- in the information set of β whenever $(r - c_S)q_+ > (r - c_S)q_- - so_S(q_+ - q_-)$ which always holds, too. Likewise, we observe that for each decision node in R, the utility does not improve if the strategy is changed, therefore the strategy $s_{G1}^{s2} = ((q_-, q_+), (q_+, q_-))$ is also a separating equilibrium no matter it implies to lie and not to follow the signal.

Pooling signals: In this part, we analyze two strategies that imply that the sender always sends the same signal, therefore the receiver must build a belief based on the demand probability distribution. When the receiver observes pooling signals, he knows the information set associated with the signal and based on this information, must determine his best response. It is important to note that for this case, supplier actions at the information set which is not associated with the received signal are not relevant to the final outcome, so we can propose a strategy that implies convenient actions to construct an equilibrium.

The strategy $s_R = (q_-, q_-)$ represents the case where the sender lies sometimes. The extensive form game corresponding to this case is as follows:

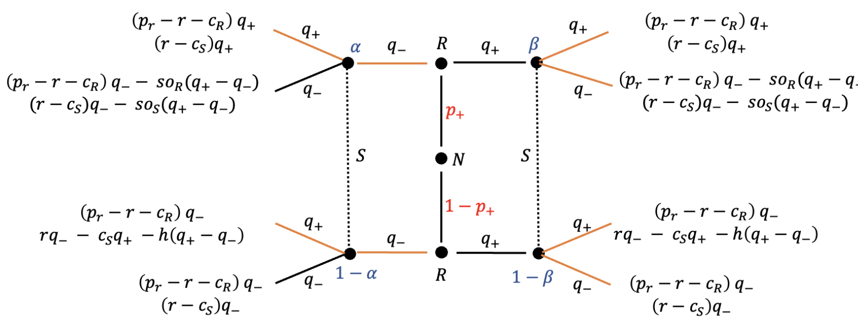

Figure 5.4: G_1 under *low* pooling signals.

We note that for this case $\alpha = p_+$ and that S will choose q_+ in that information set whenever the following inequality holds:

$$(p_+)[(r - c_S)q_+] + (1 - p_+)[rq_- - c_S q_+ - h(q_+ - q_-)] > (p_+)[(r - c_S)q_- - so_S(q_+ - q_-)] + (1 - p_+)[(r - c_S)q_-]. \quad (5.4)$$

From Expression 5.4 we can conclude that it will depend entirely on the value of the variables whether q_+ is a better response for the information set of α. Also when Expression 5.3 is true we have that $s_{G1}^{p1} = ((q_-, q_-), (q_+, (aq_+, (1-a)q_-)))$

for all $a \in [0,1]$, is a pooling equilibrium. On the other hand, if the previous inequality does not hold, we have that $s_{G1}^{p2} = ((q_-, q_-), (q_-, q_-))$ is a pooling equilibrium. In summary, it turns out that, under this scheme, to lie is an equilibrium strategy sometimes, even if the signal is followed or not depending on the values of the variables.

Another case occurs when R sends the pooling signal $s_R = (q_+, q_+)$, that is, when he always tells the supplier that the market demand is high. We have the graphical representation of this situation as follows:

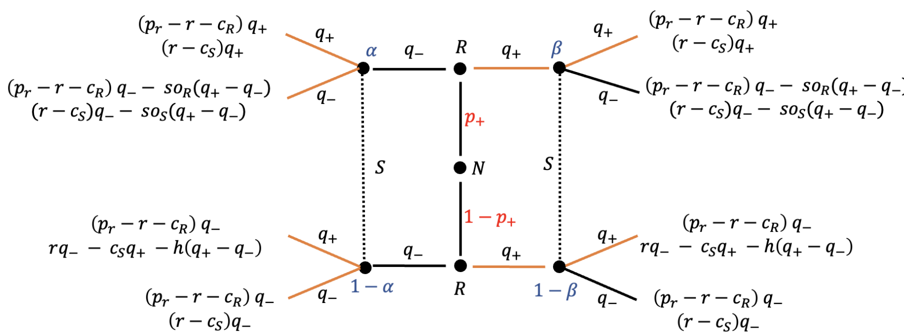

Figure 5.5: G_1 under *high* pooling signals.

For this case we observe that $\beta = p_+$ and that S will choose q_+ in α's information set, allways that next inequality holds:

$$(p_+)[(r - c_S)q_+] + (1 - p_+)[rq_- - c_S q_+ - h(q_+ - q_-)] > \\ (p_+)[(r - c_S)q_- - so_S(q_+ - q_-)] + (1 - p_+)[(r - c_S)p_-]. \quad (5.5)$$

From Expression 5.5 we can deduce the same conclusions that came from Expression 5.4. Thus, we conclude that $s_{G1}^{a1} = ((q_+, q_+), ((aq_+, (1-a)q_-), q_+))$ is a pooling equilibrium if 5.5 holds. If 5.5 does not hold, then $s_{G1}^{a3} = ((q_+, q_+), (q_-, q_-))$ is pooling equilibrium. □

To conclude this section, we highlight that under this scheme the game shows several rational ways to behave in this situation. Previous rules were proposed based on the work of Slimani, et. al. (2014) and common policies in the relationship between big corporations and their commercial partners. Shown rational ways of acting do not necessarily imply sending correct signals and or following them. That result is not entirely desired since it implies that rational behaviour does not necessarily go hand in hand with cooperation between agents. These conclusions motivate us to propose in the following section a policy modification under which we can assure rational behavior and cooperative behavior side by side.

5.1.1 A modified signaling game to model a supply chain.

In this section, we will propose different rules to govern the purchase, return policy, commercialization quantity commitment policies, and stock-outs costs. Our proposal is clearly adapted to the simplified situation that we have worked on throughout this chapter, which was mainly thought to propose a way that ensures that rational behaviour is only possible through cooperative behaviour.

The first modification will be made to the commercialization unit cost. In Section 5.1 we establish that the retailer has to receive the entire production lot from the supplier to commercialize it no matter the signal he send. In this new proposal, we have two possible ways to carry out those costs depending on the sending signal. First: if the retailer sends the signal q_- and this is true, but the supplier decides to produce q_+, that is to say, that it does not follow the signal, then, the marketing cost for the retailer will be $c_R q_-$ which implies that the retailer does not have to market large lots when he knows that there will be leftover pieces and he communicated this information to the supplier. Second: if the retailer sends the false signal q_+ and the supplier decides to follow it, then the retailer's marketing cost will remain $c_R q_-$ but will fully absorb the unit leftover cost h as a consequence of lying.

Regarding the stock-out cost, we propose that when the retailer sends the true signal q_+, but the supplier decides to produce q_-, then the supplier has to handle stock-out costs due to not having followed the signal. On the other hand, if the retailer sends the false signal q_- and the supplier follows this signal, then the stock-out cost will be carried by the retailer as a consequence of lying.

Finally, for the cases in which the retailer lies and the provider does not follow the signal, the retailer will be penalized with a loss of trust cost denoted by $e > 0$. This case can be explained as a situation where an agent requests information that he already knows with the only purpose of finding out if the other agent is cooperating or not. Thus, the signaling game associated with our proposal is given by the following extensive form game:

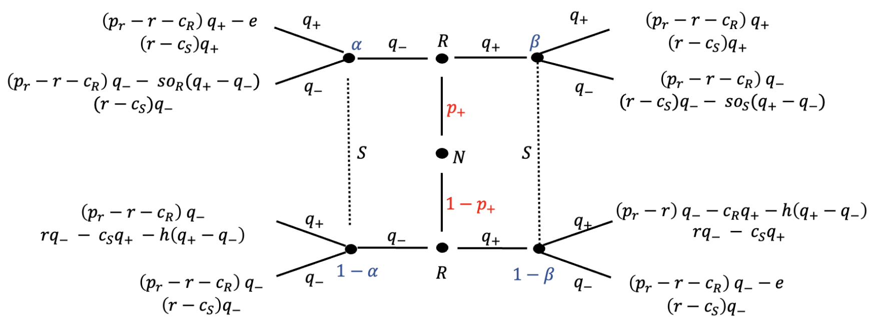

Figure 5.6: Signaling game G_2

Next, we will state our second Theorem.

Theorem 6. *For the signaling game G_2 the strategy*

$$s_{G2}^{s1} = ((q_+, q_-), (q_-, q_+))$$

it is the only separating equilibrium in the game. Furthermore, if

$$e > (pr - r - c_R)(q_+ - q_-)$$

there are no pooling equilibrium.

Proof. This proof will be divided in two parts according to the signal types and will show all the cases in an exhaustive way.

Separating signals: As in Section 5.1, we will analyze through the corresponding scheme what implications it has for the agents that the retailer always tells the truth, that is, sends the signal $s_R = (q_+, q_-)$:

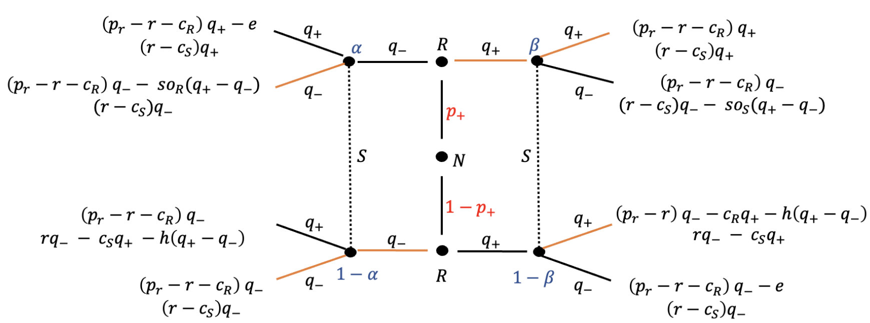

Figure 5.7: G_2 under *true* separating signals.

We note that for this case $\alpha = 0$ and $\beta = 1$, then S will choose q_- in α's information set, because $(r - c_S)q_- > rq_- - c_Sq_+ - h(q_+ - q_-)$ always holds regardless the value of the variables. And S will choose q_+ in β's information set whenever $(r - c_S)q_+ > (r - c_S)q_- - so_S(q_+ - q_-)$ holds, which always happens. Therefore $s^{s1}_{G2} = ((q_+, q_-), (q_-, q_+))$, is a separating equilibrium.

On the other hand, when the retailer decides to send the signal $s_R = (q_-, q_+)$ we get the following game structure:

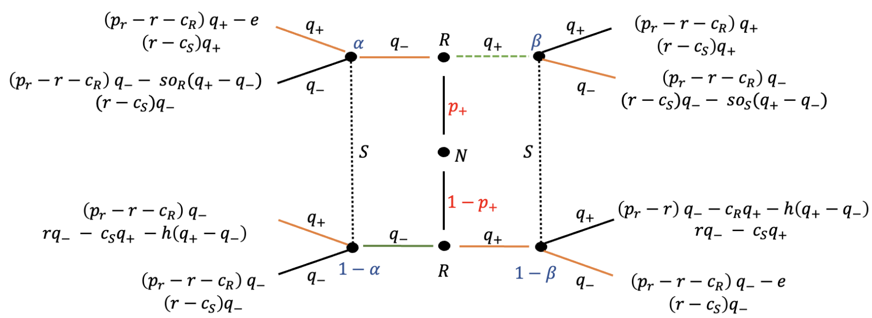

Figure 5.8: G_2 under *false* separating signals.

We can see that $\alpha = 1$ and $\beta = 0$ which implies that q_+ is the best answer for S in α 's information set given that $(r - c_S)q_+ > (r - c_S)q_-$ is always true. And, S will choose q_- in β's information set whenever $(r - c_S)q_- > rq_- - c_Sq_+$ which is also always true. On the other hand, we observe that it is convenient for R to change its strategy in the node associated to $(1 - p_+)$. He would improve his utility, if changes his action as highlighted in Figure 5.8 with the solid green line. Additionally, if $e > (p_r - r - c_R)(q_+ - q_-)$, we have that if R changes the strategy associated with the p_+'s node he would choose his best answer. Thus, we conclude that there is no equilibrium associated with the lying separating signal.

Pooling signals: In this part, we will analyze the cases where the retailer decides to send the same signal without considering the true market demand. Thus, for the signal $s_R = (q_-, q_-)$ we have the Figure 5.9 game structure. We see that $\alpha = p_+$ and so, S will choose q_+ in α's information set, always that we have:

$$(p_+)[(r - c_S)q_+] + (1 - p_+)[rq_- - c_Sq_+ - h(q_+ - q_-)] > (r - c_S)q_-. \quad (5.6)$$

From Expression 5.6 we can see that, depending on the value of the variables, there will be situations where S's best response is given by q_+ at α's information set and this case is indicated by the purple lines. Also, we can note that if $e > (q_+ - q_-)(p_r - r - c_R)$ then R would not be choosing his best response in

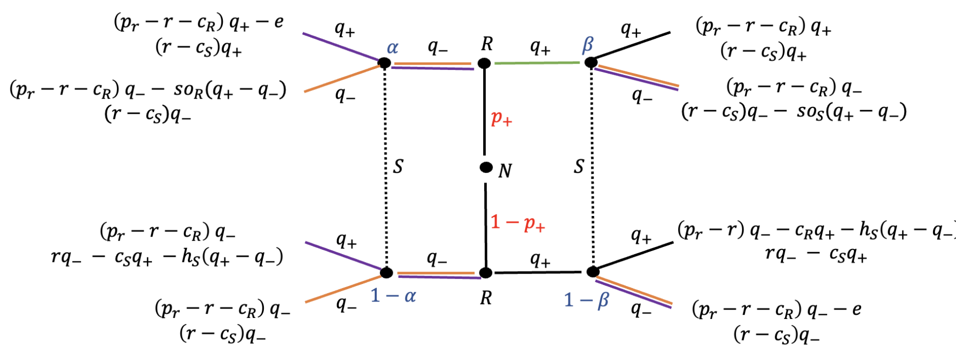

Figure 5.9: G_2 under *low* pooling signals.

p_+'s node. Then, considering the case where q_- is the best response for S in α's information set then R would not be choosing rationally at the node associated with p_+ as is indicated by the green line. Thus, we show that there are no equilibria associated with the signal $s_R = (q_-, q_-)$.

Finally, we have the case associated to the pooling signal $s_R = (q_+, q_+)$ and we have the following game structure:

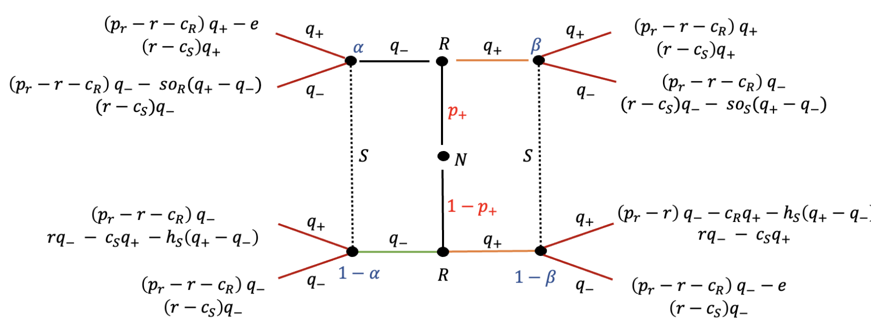

Figure 5.10: G_2 under *high* pooling signals.

For this case we have that $\beta = p_+$ and S will choose q_+ in β's information set as long as the next expression holds:

$$(p_+)[(r - c_S)q_+] + (1 - p_+)[rq_- - c_S q_+] > \\ (p_+)[(r - c_S)q_- - so_S(q_+ - q_-)] + (1 - p_+)[(r - c_S)q_-]. \quad (5.7)$$

From Expression 5.7 we can see that, depending on the value of the variables, there will be occasions where a q_+ is the best response for S in β's information set. However, whatever S's best response is in β's information set and for any strategy in α's information set, as illustrated by the red lines, we will have that R's best response always imply to choose q_- at the node associated with $(1 - p_+)$, as indicated by the green line. Thus, we show that it is not possible to construct an equilibrium from the signal $s_R = (q_+, q_+)$. □

Finally, we can conclude that with proposed policies, there is only one rational way to act that guarantees cooperation between the agents.

5.1.2 Conclusions.

The main contribution of this section is focused on proposing signal games as a tool to analyze information exchange through different supply chain models that are proposed by the literature and inquire about the mainspring that causes non-cooperative behaviour. The most interesting models for studying with signaling games are those where it is possible to model some interesting variable (like market demand) as a random variable. This idea aims to the objective of driving the agents with better information to share it with the other agents which have no so well information quality.

When the policies of production, return, repurchase, etc. are focused on the main objective of reducing general costs, a kind of interaction between agents is encouraged to favor the performance of that optimization process. But in most cases, this process is made without taking into account the important role of the information flow and its quality to achieve these purposes. This is why we are interested in analyzing different generalizations of our own model and comparing it with the most common models that are proposed in terms of supply chains analysis.

Now, we will discuss some direct and unsolved generalizations of our model that will imply a reconsideration of the agents' strategy spaces. For example, consider that the strategy space of S is given by $S_S = [q_-, q_+]$. In addition, R's strategies space could consider more complex signals, such as a continuous interval, which would represent a higher difficult challenge for theoretical analysis due to the difficulty that involves establishing a classification of the signals and their interpretation.

Another possible way to see our proposal is to consider that the retailer only obtains partial information regarding the market demand, that is, the information that he can obtain is only about the distribution of demand. This fact implies in terms of our original problem, that the true demanded quantity will reveal until the final stage of the game when the agents have already chosen their actions.

Modifications proposed in the last paragraph consider more realistic information structures than real-world supply chain problems shown in this chapter. In addition, they highlight the importance of taking into account real incentives in negotiation protocols to achieve cooperation among agents.

Finally, we remark on the importance of having a global vision to test cooperation behavior in models that originally were made with different proposes like optimization costs or other similar aims. The simplicity of our model lets us identify variables that influence rational and cooperative behaviour and focus on the importance of the information exchange quality with a particularly simple but useful tool.

5.2 An application of auctions as negotiation mechanism in a supply chain.

In this section, we will propose a situation that involves one supplier and two retailers bargaining for many identical items. It is supposed that the supplier cannot supply the entire demand, so he always carries one client, at least, with partially or completely unsatisfied requirements.

To clarify this problem, let us propose the next notation:

- $Q \in \mathbb{N}$: the total production.
- $q_i \in \{Q+1-q_{-i}, ..., Q\}$: is the quantity that agent i requires from the supplier.
- m_i: is agent i's allocation.
- v_i: is agent i's private and independent valuation for one item.
- i^+: is the bidder with the highest bid.
- i^-: is the bidder with the lowest bid.

Thus, in terms of our notation, the supplier always faces the problem that $q_1 + q_2 > Q$. For this case, we notice that one form of commercializing a complete lot of items, is selling each item to a given market price, say $\hat{p}$, and making an arbitrary distribution of the Q items between both agents. This procedure results in a situation where the supplier always obtains an income equal to $Q \cdot \hat{p}$ and always faces the problem of having at least one unsatisfied commercial partner.

We consider an alternative mechanism that assigns the items through a sealed-bid first-price auction mechanism. This proposal consists on asking the retailers to send a bid containing two information pieces: first, the true requirement, and second, the unitary price that the bidder is able to pay according to his complete or partial requirement. Let us represent a bid by a two-entry vector, that is, $B_i = (q_i, b_i)$ where the first entry represents the true requirement and the second entry represents the unitary bid that bidders are willing to pay for one item. If, at least, one of the bids is bigger than $\hat{p}$ then, the items will be assigned according to the magnitude of the unitary bids, that is, the supplier will satisfy bidder i^+'s requirement and will assign the rest, $Q - q_{i^+}$, to bidder i^-. Thus, bidder i^+ will pay $(b_{i^+}q_{i^+})$ and bidder i^- will pay $\hat{p}(Q - q_{i^+})$. If both bids are smaller than $\hat{p}$ the assignation rule will be the same but each bidder pays $\hat{p}$ as a unitary price. For example, suppose that we have a problem where $Q = 10$, $q_1 = 7$, $q_2 = 5$, and bidders send the following bids: $B_1 = (7, 1)$ and $B_2 = (5, 2)$. For this case, we have $m_1 = 5$, $m_2 = 5$ and the seller's revenue is $(5 \cdot \hat{p}) + (5 \cdot 2)$. Notice that in this first approach, we are considering that bidders must reveal their true requirements. This supposition is for maintaining simple our mechanism and keeping it similar to the normal way of bargaining where retailers always ask for the number of items that they require. Thus, the strategic behaviour and the assignation rule only concern the magnitude of the unitary bid.

Additionally, our model considers a stock-out unitary cost that affects the retailer according to the shortage of items when this is the case, which means that the assignation process gives the retailer fewer items that require. This cost is proposed to model a possible loss that a retailer can face because of the negative implications caused by the lack of items for satisfying demand, for example, the loss of clients, a sale goal not achieved, an overall decline in sales and revenue, etc. Denoting the stock-out unitary cost by α we have that in the previous example Agent 1 would have a total stock-out cost given by 2α.

We suppose that Q is common knowledge but the opponent's requirement is unknown. For modeling, this lack of information we propose to analyze the problem from Bidder 1 perspective and define $Q_2|Q_1 = q_1$ as a discrete random variable associated to Bidder 2's requirement. The lower limit of the support of our random variable support is given by $Q + 1 - q_1$ since in case of losing the auction, Bidder 1's biggest possible assignation would be equal to $q_1 - 1$ and the lowest would be zero, thus the upper limit would be given by Q. To keep the analysis simple, we propose that $Q_2|Q_1 = q_1 \sim \mathcal{U}\{Q + 1 - q_1, Q\}$. Thus, for all $q_2 \in \{Q + 1 - q_1, ..., Q\}$, $P(Q_2|Q_1 = q_1 = q_2) = \frac{1}{q_1}$.

Modeling the previous situation as a symmetric Bayesian game, v_i is agent i's private independent value associated with the valuation for one item in the compact set $[\hat{p}, \bar{v}]$ where F and f are the distribution and density functions respectively. Notice that this model considers that retailers' valuation for one item is always greater than $\hat{p}$. This assumption is made to establish that retailers are willing to pay more than the price $\hat{p}$ since they are interested on avoiding stockout costs. Then, if we define $K|Q_1 = q_1 \equiv (Q - m_2)$ we can notice that $K|Q_1 = q_1 \sim \mathcal{U}\{0, q_1 - 1\}$ and that for all $k \in \{0, ..., q_1 - 1\}$ we have that $P(K|Q_1 = q_1 = k) = \frac{1}{q_1}$. Notice that the assignation rule only concerns b_i magnitude having thus that agent 1's expected utility is given by:

$$\pi(b_1) = [q_1(v_1 - b_1)]P(b_2 < b_1) + \frac{1}{q_1}\left(\sum_{k=0}^{q_1-1}[k(v_1 - \hat{p}) - \alpha(q_1 - k)]\right)P(b_2 > b_1). \quad (5.8)$$

Considering that agents' bids is defined by $b_i = b(v_i)$ an increasing function that represents agent 1's unitary bid, that agent 2 bids according to his own valuation v_2 and that agent one has to figure out his best response, we have the next expression:

$$\pi(x) = [q_1(v_1 - b(x))]P(b(v_2) < b(x)) + \frac{1}{q_1}\left(\sum_{k=0}^{q_1-1}[k(v_1 - \hat{p}) - \alpha(q_1 - k)]\right)P(b(v_2) > b(x)). \quad (5.9)$$

The previous construction let us state the next theorem:

Theorem 7. *For the case of only one supplier and two retailers, the first-price sealed-bid auction where $q_1 + q_2 > Q$, the auctioning items are considered identical, the individual valuation is a private independent value between $[\hat{p}, \bar{v}]$, $\hat{p}$ is the item market price and the bid strategy is given by $B_i = (q_i, b(\cdot))$,*

$$b^*(v_i) = \begin{cases} \left(\frac{q_i+1}{2q_i}\right)\left(v_i + \alpha - \frac{\int_{\hat{p}}^{v_i} F(x)dx}{F(v_i)}\right) + \left(\frac{q_i-1}{2q_i}\right)\hat{p}, & \text{if } v_i > \hat{p}\ , \\ \hat{p} + \alpha\left(\frac{q_i+1}{2q_i}\right), & \text{if } v_i = \hat{p}. \end{cases} \quad (5.10)$$

is a SBNE.

Proof. Given that $b(\cdot)$ is an increasing function, we can take $b^{-1}(\cdot)$ in both sides of the probabilistic terms and considering that $P(v_2 < x) = F(x)$ we obtain:

$$\pi(x) = [q_1(v_1 - b(x))]F(x) + \frac{1}{q_1}\left(\sum_{k=0}^{q_1-1}[k(v_1 - \hat{p}) - \alpha(q_1 - k)]\right)(1 - F(x)). \quad (5.11)$$

Isolating $(v_1 - \hat{p})$ from the sum, simplifying and grouping some terms we obtain:

$$\pi(x) = [q_1(v_1 - b(x))]F(x) + \left((v_1 - \hat{p})\left(\frac{q_1 - 1}{2}\right) - \alpha\left(\frac{q_1 + 1}{2}\right)\right)(1 - F(x)). \quad (5.12)$$

Deriving Equation 5.12 we obtain the next expression:

$$\pi'(x) = [q_1(v_1 - b(x))]f(x) - b'(x)q_1F(x) - \left[\left(\frac{q_1 - 1}{2}\right)(v_1 - \hat{p}) - \left(\frac{q_1 + 1}{2}\right)\alpha\right] f(x). \quad (5.13)$$

Taking first-order conditions and making $x = v_1$ by symmetric equilibrium definition, we obtain the next equivalence:

$$[q_1(v_1 - b(v_1))]f(v_1) = b'(v_1)q_1F(v_1) + \left[\left(\frac{q_1 - 1}{2}\right)(v_1 - \hat{p}) - \left(\frac{q_1 + 1}{2}\right)\alpha\right] f(v_1). \quad (5.14)$$

Gruping $b(v_1)$ and $b'(v_1)$ in the left side of 5.14 and grouping v_1 and $\hat{p}$ in the right side we obtain:

$$b'(v_1)q_1F(v_1) + b(v_1)q_1f(v_1) = q_1v_1f(v_1) - \left[\left(\frac{q_1 - 1}{2}\right)(v_1 - \hat{p}) - \left(\frac{q_1 + 1}{2}\right)\alpha\right] f(v_1). \quad (5.15)$$

Simplifying some terms we have:

$$q_1[b(v_1)F(v_1)]'_{v_1} = \left[\left(\frac{q_1 + 1}{2}\right)(v_1 + \alpha) + \left(\frac{q_1 - 1}{2}\right)\hat{p}\right] f(v_1). \quad (5.16)$$

Applying the Fundamental Theorem of Calculus and isolating q_1 from the left-hand side we have the next implication:

$$b(v_1)F(v_1) = \left(\frac{q_1 + 1}{2q_1}\right)\int_{\hat{p}}^{v_1} xf(x)dx + \left[\left(\frac{q_1 + 1}{2q_1}\right)\alpha + \left(\frac{q_1 - 1}{2q_1}\right)\hat{p}\right] F(v_1). \quad (5.17)$$

Thus, we can show our candidate for an equilibrium bid strategy:

$$\begin{aligned} b^*(v_1) &= \left(\frac{q_1+1}{2q_1F(v_1)}\right)\int_{\hat{p}}^{v_1} xf(x)dx + \left(\frac{q_1+1}{2q_1}\right)\alpha + \left(\frac{q_1-1}{2q_1}\right)\hat{p} \\ &= \left(\frac{q_1+1}{2q_1}\right)\left(\frac{\int_{\hat{p}}^{v_i} xf(x)dx}{F(v_i)} + \alpha\right) + \left(\frac{q_1-1}{2q_1}\right)\hat{p}. \end{aligned} \tag{5.18}$$

Now, we are going to show that $b^*(\cdot)$ is indeed an equilibrium. Isolating $b'(v_1)q_1F(v_1)$ from Equation 5.14 and evaluating in $x \neq v_1$ we obtain the next condition:

$$\begin{aligned} b'(x)q_1F(x) = [q_1(x - b(x))]f(x) \\ - \left[\left(\frac{q_1-1}{2}\right)(x-\hat{p}) - \left(\frac{q_1+1}{2}\right)\alpha\right] f(x). \end{aligned} \tag{5.19}$$

Substituing Equation 5.19 into Equation 5.13 and simplifing some terms we obtain the next condition:

$$\pi'(x) = \left[\left(\frac{q_1+1}{2}\right)(v_1 - x)\right] f(x). \tag{5.20}$$

Let be $x < v_1$. From Equation 5.20 we observe that $\pi'(x) > 0$ and with similar reasoning, we can conclude that if $x > v_1$ then $\pi'(x) < 0$ probing that $b^*(\cdot)$ is indeed an equilibrium bid. Finally, considering that, $\int_{\hat{p}}^{v_1} xf(x)dx = v_1F(v_1) - \int_0^{v_1} F(x)dx$, we can express our equilibrium bid as follows:

$$b^*(v_1) = \left(\frac{q_1+1}{2q_1}\right)\left(v_1 + \alpha - \frac{\int_0^{v_1} F(x)dx}{F(v_1)}\right) + \left(\frac{q_1-1}{2q_1}\right)\hat{p}. \tag{5.21}$$

Now we are going to propose how to bid when $v_1 = \hat{p}$ since $b^*(\cdot)$ is not defined for this case. Notice that $\int_{\hat{p}}^{v_1} xf(x)dx > \int_{\hat{p}}^{v_1} \hat{p}f(x)dx = \hat{p}F(v_1)$ so, the infimum for $b^*(\cdot)$ would be given by:

$$\hat{p} + \alpha\left(\frac{q_1+1}{2q_1}\right) \tag{5.22}$$

So, for being consistent with our equilibrium bid we propose that when the valuation of the item was the lowest possible then, we take the maximum lower bound as a bid for this case. We are sure that this proposal is an equilibrium because from 5.12 we have that $\pi(x) = (v_1 - \hat{p})\left(\frac{q_1-1}{2}\right) - \alpha\left(\frac{q_1-1}{2q_1}\right)$ so, since $\pi(\cdot)$

does not depend on $b(\cdot)$, there is no strategic behaviour that can improve the expected utility when $v_1 = \hat{p}$.

□

Notice that the larger the stock-out cost, the greater is the equilibrium bid, even when the valuation of the object is equal to $\hat{p}$ the bid is always greater than this value. This last conclusion exhibits the individual valuation was proposed as a disaggregate variable with two components, first the private independent value v_i and, second, the stock-out cost α which is assumed as a symmetric value for the agents. Thus when $\alpha = 0$ we have the next expression as the equilibrium bid:

$$b^*(v_i) = \begin{cases} \left(\frac{q_1+1}{2q_1}\right)\left(v_i - \frac{\int_{\hat{p}}^{v_i} F(x)dx}{F(v_i)}\right) + \left(\frac{q_1-1}{2q_1}\right)\hat{p}, & \text{si } v_i > \hat{p}\ , \\ \\ \hat{p}, & \text{si } v_i = \hat{p}. \end{cases} \tag{5.23}$$

Since $\left(\frac{q_1+1}{2q_1}\right) + \left(\frac{q_1-1}{2q_1}\right) = 1$, we can see that our equilibrium bid is a kind of a convex combination between v_i and $\hat{p}$ and the term $\left(\frac{q_1+1}{2q_1}\right)\left(\frac{\int_{\hat{p}}^{v_i} F(x)dx}{F(v_i)}\right)$ is the amount of shading.

On the other hand, we have that since the maximum lowest bound is given by 5.22 we can assure that the seller's expected revenue is always greater under this mechanism than selling the items at a given market price $\hat{p}$ and allocating they in an arbitrary way.

5.2.1 Conclusions and future work

There are many possible generalizations of the proposed problem. The first possible modification could be concerned with assuming α as a symmetric cost and in that case, we can propose a unitary bidding function that depends on two criteria: the individual valuation and the personal stock-out cost. The technical limitations for this case lie in how to compare bidding functions that depend of two variables and then calculate the probabilities.

Another possibility concerns considering the first component of the bid into the allocation mechanism, what is considered the retailer's requirement as part of the strategy, it is to say, that a bidder asked the number of items according to a function of the real requirement and try to determine how many items an agent has to ask in order to maximize his expected utility given that others ask according to some criteria. The technical limitations for this case are the same that in the last paragraph.

Also, it is interesting to consider possible generalizations of our mechanism to a context for more than two bidders. One possible way to do the assignation process is only considering the unitary bid as in our previous case and ordering the unitary bids from the highest to the lower. Then, the supplier will assign the items satisfying the retailer with the highest unitary bid first, then the retailer with the second highest unitary bid, and so on until the supplier run out of the items. For example, consider the case where $Q = 12, q_1 = 4, q_2 = 5, q_3 = 6$, then if $b_1 = 3, b_2 = 2, b_3 = 1$ we have that $m_1 = 4, m_2 = 5$ and $m_3 = 3$. Notice that bidder 3's assignment was given by $Q - q_1 - q_2$. The theoretical difficulties of this case concern the hardness of the calculation of how many items a player receives when his unitary bid is the second highest or the third highest, etc. There is no deterministic form to knowing that information at most we could calculate the expected value of the number of items when a player does not have de highest unitary bid.

Finally, we like to conclude this section by mentioning the importance of our proposal for solving assignation problems by prioritizing the seller's revenue, especially in cases where a supplier can not satisfy the complete market demand. It is important to remark that, the actual tendency of many kinds of social partners, to face bargaining problems in automatic business applications from their cell phones or many-other platforms on the internet, offers the possibility to make easy the bargaining problem under our proposal. This fact keeps us interested in analyzing rational behaviour in this kind of situation.

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